A New Approach to Multiple Criteria Decision-Making Using the Dice Similarity Measure under Fermatean Fuzzy Environments

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Abstract

Many contemporary multiple criteria decision-making (MCDM) problems are rather complicated and uncertain to manage. MCDM problems can be complex because they involve making decisions based on multiple conflicting criteria, and they can be uncertain because they often involve incomplete or subjective information. This can make it difficult to determine the optimal solution to the problem. Over the last decades, tens of thousands MCDM methods have been proposed based on fuzzy sets (FSs) and intuitionistic fuzzy sets (IFSs). In this paper, we propose a new MCDM method based on Fermatean fuzzy sets (FFSs) and improved Dice similarity measure (DSM) and generalized Dice similarity measures (GDSM) between two FFSs with completely unknown weights of criteria. When a decision matrix is given, we calculate the weights of criteria using a normalized entropy measure while the weights of criteria are not given by the decision-maker. Then, we use the proposed improved DSM and GDSM between two FFSs that take the hesitancy degree of elements of FFSs into account and develop a new MCDM method. Finally, we use the values of the proposed improved DSM and GDSM between two FFSs to get the preference order of the alternatives. The proposed method can overcome the drawbacks and limitations of some existing methods that they cannot get the preference order of the alternatives under Fermatean fuzzy (FF) environments.

Keywords: Dice Similarity Measure (DSM), Fermatean Fuzzy Sets (FFSS), Generalized Dice Similarity Measures (GDSM), Multiple Criteria Decision-Making (MCDM), *q*th Rung Orthopair Fuzzy Set (*q*-ROF)

1 Introduction

Each organization today faces complicated and uncertain decision-making (DM) problems in real life. It can be challenging to determine the optimal solution to a problem, especially when the problem is complex or when there are many different criteria that need to be taken into account. The negative consequences are that most decision-makers are acting without a comprehensive vision of how and why uncertainties are being made. In order to solve uncertain and imprecise DM problems, Zadeh's fuzzy sets (FSs) [28] and Atanassov's intuitionistic fuzzy sets (IFSs) [1] improve our thinking on complicated and unknown DM problems. IFS is an extension of FS and is more flexible than FS. The sum of membership and non-membership of the classic IFSs is bounded by 1 while the Pythagorean fuzzy sets (PFSs) [26] sum up the squares of membership and non-membership which is bounded by 1. Therefore, PFSs can get more freedom than IFSs on uncertain and imprecise decision information. Furthermore, Fermatean fuzzy sets (FFSs) [20] sum up the cubes of membership and non-membership which is bounded by 1. Accordingly, the space of FFS's membership grades is larger than that of PFS's. In other words, the set of Fermatean membership grades can handle a higher level of uncertainty than the set of Pythagorean membership grades. Therefore, in comparison to constraints on ranges of IFSs and PFSs, FFSs remove the restriction on the representation of knowledge of membership grades and allow us to specify orthopair membership grades more space.

Many researchers contributed to various DM applications under intuitionistic fuzzy (IF) environments [3-4, 7, 14-16, 21, 24]. In [3], Baccour *et al*. applied different operators to explore comprehensively many similarity and distence measures between IFSs and made comparisons between those measures which omitted the influence of hesitancy degree for different applications. In [4], Chen and Li made a comparative analysis of existing IF entropy measures to determine objective weights and proposed a new objective entropy-based weighting method for solving multiple attribute decision-making (MADM) problems. In [7], Garg presented an IF group DM method with an improved cosine similarity measure to solve pattern recognition, medical diagnosis and investment problems. In [14], Lee *et al.* proposed a MCDM method based on a weighted similarity measure (WSM) and an extended TOPSIS method with unknown weights of criteria in order to select an appropriate sustainable and green building materials supplier in the initial stage of the supply chain (SC) under IF environments. In [15], Li *et al.* presented a weighted induced ordered weighted averaging operator based on weighted induced distance and used the induced aggregation distance operator to solve the

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DOI: 10.53106/160792642023072404003

investment selection problem through group DM process. In [16], Phochanikorn and Tan proposed an extended MCDM method under an IF environment for sustainable supplier selection based on DEMATEL and analytic network process (ANP) to identify uncertainties and interdependencies for improving sustainability in the SC. In [21], Singh and Kumar presented a DSM to solve pattern and face recognition problems under IF environments. In [24], Xia and Xu proposed an entropy and cross entropy measures combining with aggregation operators to solve group DM problems under IF environments. Some researchers contributed to various DM applications under Pythagorean fuzzy (PF) environments [23]. In [23] Wang *et al.* proposed some DSMs of PFSs and GDSMs of PFSs and applied these measures to solve multiple attribute group DM problems. Furthermore, many researchers contributed to various DM applications under Fermatean fuzzy (FF) environments [5, 8-10, 13, 18-20, 22]. In [5], Deng and Wang proposed a method combining Dempster-Shafer theory and FF entropy measure to solve MCDM problems under FF environments. In [8], Garg *et al.* proposed the DSMs and GDSMs for complex qth rung orthopair fuzzy set (q-ROF for short) and applied the proposed measures with some numerical examples related to medical diagnoses and pattern recognition. In [9], Garg *et al.* proposed some general aggregation operators based on Yager's Aggregation Operators under FF environments to apply in COVID-19 testing facility. In [10], Gül presented three MADM methods: Simple Additive Weighting (SAW), Additive Ratio Assessment (ARAS) and Vlsekriterijumska Optimizacija I Kompromisno Resenje (VIKOR), which are used to solve COVID-19 testing laboratory selection problems under FF environments and made a comparative analysis of these three methods. In [13], Jan *et al.* proposed some GDSMs and weighted generalized Dice similarity measures (WGDSMs) under q-ROF environments for the selection of the best car company. In [18], Sahoo defined different similarity measures between FFSs and applied these measures to solve group DM problems. In [19] Senapati and Yager proposed subtraction, division and Fermatean arithmetic mean operations over Fermatean fuzzy numbers (FFNs) and applied the weighted product model (WPM) to solve bridge construction selection problems. In [20], Senapati and Yager initiated FFSs to compare with PFSs and IFSs and proposed a FF TOPSIS method to handle MCDM problems. In [22], Silambarasan used the concept of Fermatean fuzzy matrices (FFM) to develop the Hamacher operations of FFM and defined and proved some algebraic properties.

In this paper, we propose a MCDM method based on improved DSM and improved GDSM of FFSs and applied these measures to solve sustainable building materials supplier selection problems with unknown weights of criteria. The contributions of this paper include: 1) we use the normalized entropy measure to determine the weights of criteria objectively while the weights of criteria were not given by the decision-maker; 2) we propose an improved DSM and an improved GDSM between two FFSs that take the indeterminacy degree of elements of FFSs into account; 3) we propose a new MCDM method based on novel improved DSM and improved GDSM between two FFSs that can overcome the drawbacks and limitations of IFSs' membership and non-membership grades.

The rest of this paper is organized as follows. In Section 2, we briefly review the definitions of IFSs [1], q-ROFs [27], PFSs [26], FFSs [20], DSM [6, 23] and GDSM [13]. In Section 3, we propose some new improved DSM/weighted Dice similarity measure (WDSM) and GDSM/WGDSM based on FFSs. In Section 4, we propose a new MCDM method based on improved WDSM or WGDSM and use two examples to compare the proposed method with Lee *et al.*'s method [14] for DM under FF environments. The conclusions are discussed in Section 5.

2 Preliminaries

In this section, we briefly review the definitions of IFSs [1], q-ROFs [27], PFSs [26], FFSs [20], DSMs [6, 23] and GDSM [13].

Definition 2.1 **[1]:** An IFS A in the universe of discourse (nonempty set) X, where $X = \{x_1, x_2, ..., x_n\}$, is an object represented in the form $A = \{<x_i, \mu_A(x_i), \upsilon_A(x_i)>| \ x_i \in X\}$, where the functions $\mu_A : X \rightarrow [0, 1]$ and $\upsilon_A : X \rightarrow [0, 1]$ are defined as the membership function of the IFS A and the non-membership function of the IFS A, respectively. $\mu_A(x_i)$ and $\upsilon_A(x_i)$ denote the degree of membership and the degree of non-membership of element x_i belonging to the IFS A, respectively, $0 \leq \mu_A(x_i) \leq 1$, $0 \leq \upsilon_A(x_i) \leq 1$, $0 \leq \mu_A(x_i) + \upsilon_A(x_i) \leq 1$ and $1 \leq i \leq n$. $\pi_A(x_i)$ is called the hesitancy degree of element x_i belonging to the IFS A, where $\pi_A(x_i) = 1 - \mu_A(x_i) - \upsilon_A(x_i)$ and $1 \leq i \leq n$.

For convenience, Xu [25] called $\alpha = (a, b)$ an intuitionistic fuzzy number (IFN) or an intuitionistic fuzzy value (IFV), where $0 \leq a \leq 1$, $0 \leq b \leq 1$, and $0 \leq a + b \leq 1$.

Definition 2.2 **[27]:** A q-ROF R in the universe of discourse X, where $X = \{x_1, x_2, ..., x_n\}$, can be represented by $R = \{<x_i, \mu_R(x_i), \upsilon_R(x_i)>| \ x_i \in X$, where μ_R indicates support for membership of x_i in A, $\mu_R : X \rightarrow [0, 1]$ and υ_R indicates support against membership of x_i in A, $\upsilon_R : X \rightarrow [0, 1]$, respectively, $0 \leq \mu_R(x_i) \leq 1$, $0 \leq \upsilon_R(x_i) \leq 1$, $0 \leq \mu_R^q(x_i) + \upsilon_R^q(x_i) \leq 1$, $q \geq 1$ and $1 \leq i \leq n$.

A q-ROF R has membership grade that is orthopair. It is clear that IFSs are q-ROFs with $q = 1$. Yager defined PFSs [26] which are q-ROFs with $q = 2$. The formal definition of PFS is shown as below.

Definition 2.3 **[26]:** Let X be a universe of discourse, where $X = \{x_1, x_2, ..., x_n\}$. A PFS P in X is an object having the form $P = \{<x_i, \alpha_P(x_i), \beta_P(x_i)>| \ x_i \in X\}$, where $\alpha_P: X \rightarrow [0, 1]$ and $\beta_P: X \rightarrow [0, 1]$, respectively, under the condition $0 \leq \alpha_P^2(x_i) + \beta_P^2(x_i) \leq 1$, for all $x_i \in X$ and $1 \leq i \leq n$. The numbers $\alpha_P(x_i)$ and $\beta_P(x_i)$ are the degree of membership and the degree of non-membership of the element x_i in the set P, where $1 \leq i \leq n$. For any PFS P and $x_i \in X$, $\pi_P(x_i) = \sqrt{1 - \alpha_P^2(x_i) - \beta_P^2(x_i)}$, where $\pi_P(x_i)$ is called the degree of indeterminacy of x_i to P and $1 \leq i \leq n$.

In [20], Senapati and Yager defined FFSs which are q-ROFs with $q = 3$. The formal definition of FFS is shown as below.

***Definition 2.4* [20]:** Let X be a universe of discourse, where $X = \{x_1, x_2, ..., x_n\}$. A FFS F in X is an object having the form $F = \{<x_i, \alpha_F(x_i), \beta_F(x_i)>| \, x_i \in X\}$, where α_F: $X \to [0, 1]$ and β_F: $X \to [0, 1]$, respectively, under the condition $0 \leq \alpha_F^3(x_i) + \beta_F^3(x_i) \leq 1$, for all $x_i \in X$ and $1 \leq i \leq n$. The numbers $\alpha_F(x_i)$ and $\beta_F(x_i)$ are the degree of membership and the degree of non-membership of the element x_i in the set F, where $1 \leq i \leq n$. For any FFS F and $x_i \in X$, $\pi_F(x_i) = \sqrt[3]{1-\alpha_F^3(x_i)-\beta_F^3(x_i)}$, where $\pi_F(x_i)$ is called the degree of indeterminacy of x_i to F and $1 \leq i \leq n$.

For simplicity, Senapati and Yager call $f = (a, b)$ an FFN, where $0 \leq a \leq 1$, $0 \leq b \leq 1$ and $0 \leq a^3 + b^3 \leq 1$. FFN extends the space of (a, b) with the constraint $0 \leq a^3 + b^3 \leq 1$. That is, the sum of membership and non-membership grades may be greater than 1. For example, if $a = 0.5$ and $b = 0.7$, then $0 \leq 0.5^3 + 0.7^3 \leq 1$. However, $0.5 + 0.7 \geq 1$ leads to the fact that there is some information that can be handled in FF environments rather than in IF environments.

***Definition 2.5* [6, 23]:** The DSM between two vectors of length n is defined as follows:

$$D(A,B)=\frac{2A\cdot B}{\|A\|_2^2+\|B\|_2^2}=\frac{2\sum_{j=1}^{n}a_jb_j}{\sum_{j=1}^{n}a_j^2+\sum_{j=1}^{n}b_j^2}, \tag{1}$$

where $A = \{a_1, a_2, ..., a_n\}$, $B = \{b_1, b_2, ..., b_n\}$, $A\cdot B$ is called the inner product of the vector A and B, and $\|A\|_2$ and $\|B\|_2$ are the Euclidean norms of A and B. When $a_j = b_j = 0$, where $1 \leq j \leq n$, we can let $D(A, B) = 0$.

The DSM satisifies the following properties [6, 23]:

(1) $0 \leq D(A, B) \leq 1$;

(2) $D(A, B) = D(B, A)$;

(3) $D(A, A) = 1$.

According to the classic definition of DSM, the DSM between two q-ROFs is defined as below.

***Definition 2.6* [13]:** The DSM between two q-ROFs R_A and R_B in the universe of discourse X, where $X = \{x_1, x_2, ..., x_n\}$, $R_A = \{<x_j, \mu_{R_A}(x_j), \upsilon_{R_A}(x_j)>| \, x_j \in X\}$, and $R_B = \{<x_j, \mu_{R_B}(x_j), \upsilon_{R_B}(x_j)>| \, x_j \in X\}$, and is defined as follows:

$$D_{q\text{-ROF}}(R_A, R_B) = \frac{1}{n}\sum_{j=1}^{n}\frac{2\left(\mu_{R_A}^q(x_j)\mu_{R_B}^q(x_j)+\upsilon_{R_A}^q(x_j)\upsilon_{R_B}^q(x_j)\right)}{\mu_{R_A}^{2q}(x_j)+\upsilon_{R_A}^{2q}(x_j)+\mu_{R_B}^{2q}(x_j)+\upsilon_{R_B}^{2q}(x_j)}, \tag{2}$$

where $0 \leq \mu_{R_A}(x_j) \leq 1$, $0 \leq \upsilon_{R_A}(x_j) \leq 1$, $0 \leq \mu_{R_A}^q(x_j) + \upsilon_{R_A}^q(x_j) \leq 1$, $0 \leq \mu_{R_B}(x_j) \leq 1$, $0 \leq \upsilon_{R_B}(x_j) \leq 1$, $0 \leq \mu_{R_B}^q(x_j) + \upsilon_{R_B}^q(x_j) \leq 1$, $q \geq 1$ and $1 \leq j \leq n$. IF $q = 3$, then $D_{q\text{-ROF}}(R_A, R_B)$ is converted to DSM between two FFSs which is defined as follows:

$$D_{FFS}(R_A, R_B) = \frac{1}{n}\sum_{j=1}^{n}\frac{2\left(\mu_{R_A}^3(x_j)\mu_{R_B}^3(x_j)+\upsilon_{R_A}^3(x_j)\upsilon_{R_B}^3(x_j)\right)}{\mu_{R_A}^{6}(x_j)+\upsilon_{R_A}^{6}(x_j)+\mu_{R_B}^{6}(x_j)+\upsilon_{R_B}^{6}(x_j)}, \tag{3}$$

where $0 \leq \mu_{R_A}(x_j) \leq 1$, $0 \leq \upsilon_{R_A}(x_j) \leq 1$, $0 \leq \mu_{R_A}^3(x_j) + \upsilon_{R_A}^3(x_j) \leq 1$, $0 \leq \mu_{R_B}(x_j) \leq 1$, $0 \leq \upsilon_{R_B}(x_j) \leq 1$, $0 \leq \mu_{R_B}^3(x_j) + \upsilon_{R_B}^3(x_j) \leq 1$ and $1 \leq j \leq n$.

***Definition 2.7* [13]:** The weighted Dice similarity measure (WDSM) between two q-ROFs R_A and R_B in the universe of discourse X, where $X = \{x_1, x_2, ..., x_n\}$, $R_A = \{<x_j, \mu_{R_A}(x_j), \upsilon_{R_A}(x_j)>| \, x_j \in X\}$ and $R_B = \{<x_j, \mu_{R_B}(x_j), \upsilon_{R_B}(x_j)>| \, x_j \in X\}$ is defined as follows:

$$WD_{q\text{-ROF}}(R_A, R_B) = \sum_{j=1}^{n}\omega_j\frac{2\left(\mu_{R_A}^q(x_j)\mu_{R_B}^q(x_j)+\upsilon_{R_A}^q(x_j)\upsilon_{R_B}^q(x_j)\right)}{\mu_{R_A}^{2q}(x_j)+\upsilon_{R_A}^{2q}(x_j)+\mu_{R_B}^{2q}(x_j)+\upsilon_{R_B}^{2q}(x_j)}, \tag{4}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector, $\sum_{j=1}^{n}\omega_j=1$, $0 \leq \mu_{R_A}(x_j) \leq 1$, $0 \leq \upsilon_{R_A}(x_j) \leq 1$, $0 \leq \mu_{R_A}^q(x_j) + \upsilon_{R_A}^q(x_j) \leq 1$, $0 \leq \mu_{R_B}(x_j) \leq 1$, $0 \leq \upsilon_{R_B}(x_j) \leq 1$, $0 \leq \mu_{R_B}^q(x_j) + \upsilon_{R_B}^q(x_j) \leq 1$, $q \geq 1$ and $1 \leq j \leq n$. IF $q = 3$, then $WD_{q\text{-ROF}}(R_A, R_B)$ is converted to WDSM between two FFSs which is defined as follows:

$$WD_{\text{FFS}}(R_A, R_B) = \sum_{j=1}^{n}\omega_j\frac{2\left(\mu_{R_A}^3(x_j)\mu_{R_B}^3(x_j)+\upsilon_{R_A}^3(x_j)\upsilon_{R_B}^3(x_j)\right)}{\mu_{R_A}^{6}(x_j)+\upsilon_{R_A}^{6}(x_j)+\mu_{R_B}^{6}(x_j)+\upsilon_{R_B}^{6}(x_j)}, \tag{5}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector, $\sum_{j=1}^{n}\omega_j=1$, $0 \leq \mu_{R_A}(x_j) \leq 1$, $0 \leq \upsilon_{R_A}(x_j) \leq 1$, $0 \leq \mu_{R_A}^3(x_j) + \upsilon_{R_A}^3(x_j) \leq 1$, $0 \leq \mu_{R_B}(x_j) \leq 1$, $0 \leq \upsilon_{R_B}(x_j) \leq 1$, $0 \leq \mu_{R_B}^3(x_j) + \upsilon_{R_B}^3(x_j) \leq 1$ and $1 \leq j \leq n$.

***Definition 2.8* [13]:** The GDSM between two q-ROFs R_A and R_B in the universe of discourse X, where $X = \{x_1, x_2, ..., x_n\}$, $R_A = \{<x_j, \mu_{R_A}(x_j), \upsilon_{R_A}(x_j)>| \, x_j \in X\}$ and $R_B = \{<x_j, \mu_{R_B}(x_j), \upsilon_{R_B}(x_j)>| \, x_j \in X\}$, is defined as follows:

$$GD_{q\text{-ROF}}(R_A, R_B) = \frac{1}{n}\sum_{j=1}^{n}\frac{\mu_{R_A}^q(x_j)\mu_{R_B}^q(x_j)+\upsilon_{R_A}^q(x_j)\upsilon_{R_B}^q(x_j)}{\rho\left(\mu_{R_A}^{2q}(x_j)+\upsilon_{R_A}^{2q}(x_j)\right)+(1-\rho)\left(\mu_{R_B}^{2q}(x_j)+\upsilon_{R_B}^{2q}(x_j)\right)}, \tag{6}$$

where $0 \leq \mu_{R_A}(x_j) \leq 1$, $0 \leq \upsilon_{R_A}(x_j) \leq 1$, $0 \leq \mu_{R_A}^q(x_j) + \upsilon_{R_A}^q(x_j) \leq 1$, $0 \leq \mu_{R_B}(x_j) \leq 1$, $0 \leq \upsilon_{R_B}(x_j) \leq 1$, $0 \leq \mu_{R_B}^q(x_j) + \upsilon_{R_B}^q(x_j) \leq 1$, $0 \leq \rho \leq 1$, $q \geq 1$ and $1 \leq j \leq n$. IF $q = 3$, then $GD_{q\text{-ROF}}(R_A, R_B)$ is converted to GDSM between two FFSs which is defined as follows:

$GD_{FFS}(R_A, R_B) =$

$$\frac{1}{n}\sum_{j=1}^{n}\frac{\mu_{R_A}^3(x_j)\mu_{R_B}^3(x_j)+\upsilon_{R_A}^3(x_j)\upsilon_{R_B}^3(x_j)}{\rho\left(\mu_{R_A}^6(x_j)+\upsilon_{R_A}^6(x_j)\right)+(1-\rho)\left(\mu_{R_B}^6(x_j)+\upsilon_{R_B}^6(x_j)\right)}, \quad \textbf{(7)}$$

where $0 \le \mu_{R_A}(x_j) \le 1$, $0 \le \upsilon_{R_A}(x_j) \le 1$, $0 \le \mu_{R_A}^3(x_j)+\upsilon_{R_A}^3(x_j) \le 1$, $0 \le \mu_{R_B}(x_j) \le 1$, $0 \le \upsilon_{R_B}(x_j) \le 1$, $0 \le \mu_{R_B}^3(x_j)+\upsilon_{R_B}^3(x_j) \le 1$, $0 \le \rho \le 1$ and $1 \le j \le n$.

IF $\rho = 0.5$, then the GDSM between two q-ROFs can be reduced to DSM between two q-ROFs shown as below:

$$\begin{aligned}
&GD_{q\text{-ROF}}(R_A, R_B)\\
&=\frac{1}{n}\sum_{j=1}^{n}\frac{\mu_{R_A}^q(x_j)\mu_{R_B}^q(x_j)+\upsilon_{R_A}^q(x_j)\upsilon_{R_B}^q(x_j)}{\rho\left(\mu_{R_A}^{2q}(x_j)+\upsilon_{R_A}^{2q}(x_j)\right)+(1-\rho)\left(\mu_{R_B}^{2q}(x_j)+\upsilon_{R_B}^{2q}(x_j)\right)},\\
&=\frac{1}{n}\sum_{j=1}^{n}\frac{\mu_{R_A}^q(x_j)\mu_{R_B}^q(x_j)+\upsilon_{R_A}^q(x_j)\upsilon_{R_B}^q(x_j)}{0.5\left(\mu_{R_A}^{2q}(x_j)+\upsilon_{R_A}^{2q}(x_j)\right)+0.5\left(\mu_{R_B}^{2q}(x_j)+\upsilon_{R_B}^{2q}(x_j)\right)},\\
&=\frac{1}{n}\sum_{j=1}^{n}\frac{2\left(\mu_{R_A}^q(x_j)\mu_{R_B}^q(x_j)+\upsilon_{R_A}^q(x_j)\upsilon_{R_B}^q(x_j)\right)}{\mu_{R_A}^{2q}(x_j)+\upsilon_{R_A}^{2q}(x_j)+\mu_{R_B}^{2q}(x_j)+\upsilon_{R_B}^{2q}(x_j)},\\
&=D_{q-ROF}\left(R_A, R_B\right).
\end{aligned}$$

Definition 2.9 **[13]:** The WGDSM between two q-ROFs R_A and R_B in the universe of discourse X, where $X=\{x_1, x_2, \ldots, x_n\}$, $R_A = \{<x_j, \mu_{R_A}(x_j), \upsilon_{R_A}(x_j)>|\ x_j \in X\}$ and $R_B = \{<x_j, \mu_{R_B}(x_j), \upsilon_{R_B}(x_j)>|\ x_j \in X\}$ is defined as follows:

$WGD_{q\text{-ROF}}(R_A, R_B) =$

$$\sum_{j=1}^{n}\omega_j\frac{\mu_{R_A}^q(x_j)\mu_{R_B}^q(x_j)+\upsilon_{R_A}^q(x_j)\upsilon_{R_B}^q(x_j)}{\rho\left(\mu_{R_A}^{2q}(x_j)+\upsilon_{R_A}^{2q}(x_j)\right)+(1-\rho)\left(\mu_{R_B}^{2q}(x_j)+\upsilon_{R_B}^{2q}(x_j)\right)}, \quad \textbf{(8)}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector, $\sum_{j=1}^{n}\omega_j = 1$, $0 \le \mu_{R_A}(x_j) \le 1$, $0 \le \upsilon_{R_A}(x_j) \le 1$, $0 \le \mu_{R_A}^q(x_j)+\upsilon_{R_A}^q(x_j) \le 1$, $0 \le \mu_{R_B}(x_j) \le 1$, $0 \le \upsilon_{R_B}(x_j) \le 1$, $0 \le \mu_{R_B}^q(x_j)+\upsilon_{R_B}^q(x_j) \le 1$, $0 \le \rho \le 1$, $q \ge 1$ and $1 \le j \le n$. IF $q = 3$, then $WGD_{q-\mathrm{ROF}}(R_A, R_B)$ is converted to WGDSM between two FFSs which is defined as follows:

$WGD_{FFS}(R_A, R_B) =$

$$\sum_{j=1}^{n}\omega_j\frac{\mu_{R_A}^3(x_j)\mu_{R_B}^3(x_j)+\upsilon_{R_A}^3(x_j)\upsilon_{R_B}^3(x_j)}{\rho\left(\mu_{R_A}^6(x_j)+\upsilon_{R_A}^6(x_j)\right)+(1-\rho)\left(\mu_{R_B}^6(x_j)+\upsilon_{R_B}^6(x_j)\right)}, \quad \textbf{(9)}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector, $\sum_{j=1}^{n}\omega_j = 1$, $0 \le \mu_{R_A}(x_j) \le 1$, $0 \le \upsilon_{R_A}(x_j) \le 1$, $0 \le \mu_{R_A}^3(x_j)+\upsilon_{R_A}^3(x_j) \le 1$, $0 \le \mu_{R_B}(x_j) \le 1$, $0 \le \upsilon_{R_B}(x_j) \le 1$, $0 \le \mu_{R_B}^3(x_j)+\upsilon_{R_B}^3(x_j) \le 1$, $0 \le \rho \le 1$ and $1 \le j \le n$.

Under some circumstances, we cannot distinguish between q-ROFs calculated from the measures with Eq. (2), Eq. (3), Eq. (6) and Eq. (7). The counter case is depicted by the following example.

***Example 2.1*:** Let A = (0.400, 0.200), B = (0.252, 0.675) and C = (0.524, 0.850) be three q-ROFs with q = 3 (that is FFNs). Since $D_{FFS}(A, B) = D_{FFS}(A, C) = 0.035$, we cannot distinguish the three FFNs A, B and C.

3 The Proposed Improved DSM and GDSM between Two FFSs

The following additional and geometric operational laws proposed in [11] and [12] which take the interactions into consideration between membership grades and non-membership grades of different IFSs in order to overcome Attanassov's operational laws [2] due to the fact that the operational laws in [2] do not take the interactions into consideration between membership grades and non-membership grades of different IFSs.

Definition 3.1 **[11-12]:** Let $A = <\mu_A, \upsilon_A>$ and $B = <\mu_B, \upsilon_B>$ be two IFNs, then addition and multiplication operations are defined as below.

(1) $A \oplus B = \left\langle 1-(1-\mu_A)(1-\mu_B), (1-\mu_A)(1-\mu_B)-\pi_A\pi_B \right\rangle$, **(10)**

(2) $A \otimes B = \left\langle (1-\upsilon_A)(1-\upsilon_B)-\pi_A\pi_B, 1-(1-\upsilon_A)(1-\upsilon_B) \right\rangle$, **(11)**

where $\pi_A = 1-\mu_A-\upsilon_A$ and $\pi_B = 1-\mu_B-\upsilon_B$.

Based on Eq. (10) and Eq. (11), we define improved DSM and GDSM between two FFSs shown as below.

***Definition 3.2*:** The improved DSM between two FFSs F_A and F_B in the universe of discourse X, where $X = \{x_1, x_2, \ldots, x_n\}$, $F_A = \{<x_j, \alpha_{F_A}(x_j), \beta_{F_A}(x_j)>|\ x_j \in X\}$, and $F_B = \{<x_j, \alpha_{F_B}(x_j), \beta_{F_B}(x_j)>|\ x_j \in X\}$ is defined as follows:

$$ID_{FFS}(F_A, F_B) = \frac{1}{n}\sum_{j=1}^{n}\frac{2C_{FFS}(F_{A_j}, F_{B_j})}{T_{FFS}(F_{A_j})+T_{FFS}(F_{B_j})}, \quad \textbf{(12)}$$

where

$$\begin{aligned}
&C_{FFS}(F_{A_j}, F_{B_j}) = [\left(1-\alpha_{F_A}^3(x_j)\right)\left(1-\alpha_{F_B}^3(x_j)\right)+\\
&\left(1-\beta_{F_A}^3(x_j)\right)\left(1-\beta_{F_B}^3(x_j)\right)+\pi_{F_A}^3(x_j)\pi_{F_B}^3(x_j)],\\
&T_{FFS}(F_{A_j}) = [(1-\alpha_{F_A}^3(x_j))^2+(1-\beta_{F_A}^3(x_j))^2+\pi_{F_A}^6(x_j)],\\
&T_{FFS}(F_{B_j}) = [(1-\alpha_{F_B}^3(x_j))^2+(1-\beta_{F_B}^3(x_j))^2+\pi_{F_B}^6(x_j)],\\
&0 \le \alpha_{F_A}(x_j) \le 1,\ 0 \le \beta_{F_A}(x_j) \le 1,\\
&0 \le \alpha_{F_A}^3(x_j)+\beta_{F_A}^3(x_j) \le 1,\\
&\pi_{F_A}(x_j) = \sqrt[3]{1-\alpha_{F_A}^3(x_j)-\beta_{F_A}^3(x_j)},\\
&0 \le \alpha_{F_B}(x_j) \le 1,\ 0 \le \beta_{F_B}(x_j) \le 1,\\
&0 \le \alpha_{F_B}^3(x_j)+\beta_{F_B}^3(x_j) \le 1,\\
&\pi_{F_B}(x_j) = \sqrt[3]{1-\alpha_{F_B}^3(x_j)-\beta_{F_B}^3(x_j)},\ \text{and } 1 \le j \le n.
\end{aligned}$$

The aforementioned $ID_{FFS}(F_A, F_B)$ also satisfies the following properties:

(1) $0 \le ID_{FFS}(F_A, F_B) \le 1$;

(2) $ID_{FFS}(F_A, F_B) = ID_{FFS}(F_B, F_A)$;

(3) $ID_{FFS}(F_A, F_A) = 1$;

where $F_A = \{<x_j, \alpha_{F_A}(x_j), \beta_{F_A}(x_j)> \mid x_j \in X\}$, $F_B = \{<x_j, \alpha_{F_B}(x_j), \beta_{F_B}(x_j)> \mid x_j \in X\}$, and $X = \{x_1, x_2, \ldots, x_n\}$.

Proof.

(1) It is quite obvious that $ID_{FFS}(F_A, F_B) \ge 0$. According to DSM's property (1), the two vectors A and B must satisfies $2\sum_{j=1}^{n} a_j b_j \le \sum_{j=1}^{n} a_j^2 + \sum_{j=1}^{n} b_j^2$, where $A = \{a_1, a_2, \ldots, a_n\}$, $B = \{b_1, b_2, \ldots, b_n\}$. Then

$$\begin{aligned}
&2C_{FFS}(F_{A_j}, F_{B_j}) = 2[(1-\alpha_{F_A}^3(x_j))(1-\alpha_{F_B}^3(x_j)) + \\
&\left(1-\beta_{F_A}^3(x_j)\right)\left(1-\beta_{F_B}^3(x_j)\right) + \pi_{F_A}^3(x_j)\pi_{F_B}^3(x_j)] \\
&\le \left[\left(1-\alpha_{F_A}^3(x_j)\right)^2 + \left(1-\beta_{F_A}^3(x_j)\right)^2 + \pi_{F_A}^6(x_j)\right] + \\
&\left[\left(1-\alpha_{F_B}^3(x_j)\right)^2 + \left(1-\beta_{F_B}^3(x_j)\right)^2 + \pi_{F_B}^6(x_j)\right] \\
&= T_{FFS}(F_{A_j}) + T_{FFS}(F_{B_j}).
\end{aligned}$$

Therefore,

$$0 \le ID_{FFS}(F_A, F_B) = \frac{1}{n}\sum_{j=1}^{n} \frac{2C_{FFS}(F_{A_j}, F_{B_j})}{T_{FFS}(F_{A_j}) + T_{FFS}(F_{B_j})} \le \frac{1}{n}\sum_{j=1}^{n} 1 = 1.$$

(2)

$$\begin{aligned}
&C_{FFS}(F_{A_j}, F_{B_j}) = [(1-\alpha_{F_A}^3(x_j))(1-\alpha_{F_B}^3(x_j)) + \\
&\left(1-\beta_{F_A}^3(x_j)\right)\left(1-\beta_{F_B}^3(x_j)\right) + \pi_{F_A}^3(x_j)\pi_{F_B}^3(x_j)] \\
&= \left(1-\alpha_{F_B}^3(x_j)\right)\left(1-\alpha_{F_A}^3(x_j)\right) + \\
&\left[\left(1-\beta_{F_B}^3(x_j)\right)\left(1-\beta_{F_A}^3(x_j)\right) + \pi_{F_B}^3(x_j)\pi_{F_A}^3(x_j)\right] \\
&= C_{FFS}(F_{B_j}, F_{A_j}).
\end{aligned}$$

Therefore,

$$ID_{FFS}(F_A, F_B) = \frac{1}{n}\sum_{j=1}^{n} \frac{2C_{FFS}(F_{A_j}, F_{B_j})}{T_{FFS}(F_{A_j}) + T_{FFS}(F_{B_j})} = \frac{1}{n}\sum_{j=1}^{n} \frac{2C_{FFS}(F_{B_j}, F_{A_j})}{T_{FFS}(F_{B_j}) + T_{FFS}(F_{A_j})} = ID_{FFS}(F_B, F_A).$$

(3)

$$\begin{aligned}
&C_{FFS}(F_{A_j}, F_{A_j}) = [(1-\alpha_{F_A}^3(x_j))(1-\alpha_{F_A}^3(x_j)) + \\
&\left(1-\beta_{F_A}^3(x_j)\right)\left(1-\beta_{F_A}^3(x_j)\right) + \pi_{F_A}^3(x_j)\pi_{F_A}^3(x_j)] \\
&= \left(1-\alpha_{F_A}^3(x_j)\right)^2 + \left(1-\beta_{F_A}^3(x_j)\right)^2 + \pi_{F_A}^6(x_j)] \\
&T_{FFS}(F_{A_j}) = \left[\left(1-\alpha_{F_A}^3(x_j)\right)^2 + \left(1-\beta_{F_A}^3(x_j)\right)^2 + \pi_{F_A}^6(x_j)\right] \\
&= C_{FFS}(F_{A_j}, F_{A_j}).
\end{aligned}$$

Therefore,

$$ID_{FFS}(F_A, F_A) = \frac{1}{n}\sum_{j=1}^{n} \frac{2C_{FFS}(F_{A_j}, F_{A_j})}{T_{FFS}(F_{A_j}) + T_{FFS}(F_{A_j})} = \frac{1}{n}\sum_{j=1}^{n} \frac{2C_{FFS}(F_{A_j}, F_{A_j})}{2T_{FFS}(F_{A_j})} = \frac{1}{n}\sum_{j=1}^{n} 1 = 1. \quad \text{Q.E.D.}$$

Definition 3.3: The improved WDSM between two FFSs F_A and F_B in the universe of discourse X, where $X = \{x_1, x_2, \ldots, x_n\}$, $F_A = \{<x_j, \alpha_{F_A}(x_j), \beta_{F_A}(x_j)> \mid x_j \in X\}$, and $F_B = \{<x_j, \alpha_{F_B}(x_j), \beta_{F_B}(x_j)> \mid x_j \in X\}$ is defined as follows:

$$IWD_{FFS}(F_A, F_B) = \sum_{j=1}^{n} \omega_j \frac{2WC_{FFS}(F_{A_j}, F_{B_j})}{WT_{FFS}(F_{A_j}) + WT_{FFS}(F_{B_j})}, \quad \textbf{(13)}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector, $\sum_{j=1}^{n}\omega_{j=1}$,

$$\begin{aligned}
&WC_{FFS}(F_{A_j}, F_{B_j}) = [\left(1-\alpha_{F_A}^3(x_j)\right)\left(1-\alpha_{F_B}^3(x_j)\right) + \\
&\left(1-\beta_{F_A}^3(x_j)\right)\left(1-\beta_{F_B}^3(x_j)\right) + \pi_{F_A}^3(x_j)\pi_{F_B}^3(x_j)], \\
&WT_{FFS}(F_{A_j}) = [(1-\alpha_{F_A}^3(x_j))^2 + (1-\beta_{F_A}^3(x_j))^2 + \pi_{F_A}^6(x_j)], \\
&WT_{FFS}(F_{B_j}) = [(1-\alpha_{F_B}^3(x_j))^2 + (1-\beta_{F_B}^3(x_j))^2 + \pi_{F_B}^6(x_j)], \\
&0 \le \alpha_{F_A}(x_j) \le 1, \ 0 \le \beta_{F_A}(x_j) \le 1, \\
&0 \le \alpha_{F_A}^3(x_j) + \beta_{F_A}^3(x_j) \le 1, \\
&\pi_{F_A}(x_j) = \sqrt[3]{1-\alpha_{F_A}^3(x_j) - \beta_{F_A}^3(x_j)}, \\
&0 \le \alpha_{F_B}(x_j) \le 1, \ 0 \le \beta_{F_B}(x_j) \le 1, \\
&0 \le \alpha_{F_B}^3(x_j) + \beta_{F_B}^3(x_j) \le 1, \\
&\pi_{F_B}(x_j) = \sqrt[3]{1-\alpha_{F_B}^3(x_j) - \beta_{F_B}^3(x_j)} \ \text{ and } 1 \le j \le n.
\end{aligned}$$

If $\omega = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the improved WDSM reduces to improved DSM. That is $IWD_{FFS}(F_A, F_B) = ID_{FFS}(F_A, F_B)$.

The improved WDSM also satisfies the following properties:

(1) $0 \le IWD_{FFS}(F_A, F_B) \le 1$;

(2) $IWD_{FFS}(F_A, F_B) = IWD_{FFS}(F_B, F_A)$;

(3) $IWD_{FFS}(F_A, F_A) = 1$;

where $F_A = \{<x_j, \alpha_{F_A}(x_j), \beta_{F_A}(x_j)> \mid x_j \in X\}$, $F_B = \{<x_j, \alpha_{F_B}(x_j), \beta_{F_B}(x_j)> \mid x_j \in X\}$ and $X = \{x_1, x_2, \ldots, x_n\}$.

Proof.

(1) It is quite obvious that $IWD_{FFS}(F_A, F_B) \geq 0$. According to property (1) of $ID_{FFS}(F_A, F_B)$, the two vectors A and B must satisfies $2C_{FFS}(F_A, F_B) \leq T_{FFS}(F_A) + T_{FFS}(F_B)$, where $F_A = \{<x_j, \alpha_{F_A}(x_j), \beta_{F_A}(x_j)> \mid x_j \in X\}$, $F_B = \{<x_j, \alpha_{F_B}(x_j), \beta_{F_B}(x_j)> \mid x_j \in X\}$ and $X = \{x_1, x_2, \ldots, x_n\}$. Then

$$
\begin{aligned}
&2WC_{FFS}(F_{A_j}, F_{B_j}) = 2[\left(1-\alpha_{F_A}^3(x_j)\right)\left(1-\alpha_{F_B}^3(x_j)\right)+\\
&\left(1-\beta_{F_A}^3(x_j)\right)\left(1-\beta_{F_B}^3(x_j)\right)+\pi_{F_A}^3(x_j)\pi_{F_B}^3(x_j)],\\
&\leq [(1-\alpha_{F_A}^3(x_j))^2+(1-\beta_{F_A}^3(x_j))^2+\pi_{F_A}^6(x_j)]\\
&+[(1-\alpha_{F_B}^3(x_j))^2+(1-\beta_{F_B}^3(x_j))^2+\pi_{F_B}^6(x_j)],\\
&= WT_{FFS}(F_{A_j})+WT_{FFS}(F_{B_j}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&0 \leq IWD_{FFS}(F_A, F_B) =\\
&= \sum_{j=1}^n \omega_j \frac{2WC_{FFS}(F_{A_j}, F_{B_j})}{WT_{FFS}(F_{A_j})+WT_{FFS}(F_{B_j})} \leq\\
&\sum_{j=1}^n \omega_j = 1.
\end{aligned}
$$

(2)

$$
\begin{aligned}
&WC_{FFS}(F_{A_j}, F_{B_j}) = [(1-\alpha_{F_A}^3(x_j))(1-\alpha_{F_B}^3(x_j))+\\
&\left(1-\beta_{F_A}^3(x_j)\right)\left(1-\beta_{F_B}^3(x_j)\right)+\pi_{F_A}^3(x_j)\pi_{F_B}^3(x_j)]\\
&= \left(1-\alpha_{F_B}^3(x_j)\right)\left(1-\alpha_{F_A}^3\left(x_j\right)\right)+\\
&\left[\left(1-\beta_{F_B}^3(x_j)\right)\left(1-\beta_{F_A}^3(x_j)\right)+\pi_{F_B}^3(x_j)\pi_{F_A}^3(x_j)\right]\\
&= WC_{FFS}(F_{B_j}, F_{A_j}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&IWD_{FFS}(F_A, F_B) = \sum_{j=1}^n \omega_j \frac{2WC_{FFS}(F_{A_j}, F_{B_j})}{WT_{FFS}(F_{A_j})+WT_{FFS}(F_{B_j})} =\\
&\sum_{j=1}^n \omega_j \frac{2WC_{FFS}(F_{B_j}, F_{A_j})}{WT_{FFS}(F_{B_j})+WT_{FFS}(F_{A_j})} = IWD_{FFS}(F_B, F_A).
\end{aligned}
$$

(3)

$$
\begin{aligned}
&WC_{FFS}(F_{A_j}, F_{A_j}) = [(1-\alpha_{F_A}^3(x_j))(1-\alpha_{F_A}^3(x_j))+\\
&\left(1-\beta_{F_A}^3(x_j)\right)\left(1-\beta_{F_A}^3(x_j)\right)+\pi_{F_A}^3(x_j)\pi_{F_A}^3(x_j)]\\
&= \left[\left(1-\alpha_{F_A}^3(x_j)\right)^2+\left(1-\beta_{F_A}^3\left(x_j\right)\right)^2+\pi_{F_A}^6(x_j)\right]\\
&WT_{FFS}(F_{A_j}) = \left[\left(1-\alpha_{F_A}^3(x_j)\right)^2+\left(1-\beta_{F_A}^3(x_j)\right)^2+\pi_{F_A}^6(x_j)\right]\\
&= WC_{FFS}(F_{A_j}, F_{A_j}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&IWD_{FFS}(F_A, F_A) = \sum_{j=1}^n \omega_j \frac{2C_{FFS}(F_{A_j}, F_{A_j})}{T_{FFS}(F_{A_j})+T_{FFS}(F_{A_j})} =\\
&\sum_{j=1}^n \omega_j \frac{2C_{FFS}(F_{A_j}, F_{A_j})}{2T_{FFS}(F_{A_j})} = \sum_{j=1}^n \omega_j = 1. \quad \text{Q.E.D.}
\end{aligned}
$$

Definition 3.4: The improved GDSM between two FFSs F_A and F_B in the universe of discourse X, where $X = \{x_1, x_2, \ldots, x_n\}$, $F_A = \{<x_j, \alpha_{F_A}(x_j), \beta_{F_A}(x_j)> \mid x_j \in X\}$ and $F_B = \{<x_j, \alpha_{F_B}(x_j), \beta_{F_B}(x_j)> \mid x_j \in X\}$ is defined as follows:

$$
IGD_{FFS}(F_A, F_B) = \frac{1}{n}\sum_{j=1}^n \frac{GC_{FFS}(F_{A_j}, F_{B_j})}{\rho GT_{FFS}(F_{A_j})+(1-\rho)GT_{FFS}(F_{B_j})}, \tag{14}
$$

where

$$
\begin{aligned}
&GC_{FFS}(F_{A_j}, F_{B_j}) = [\left(1-\alpha_{F_A}^3(x_j)\right)\left(1-\alpha_{F_B}^3(x_j)\right)+\\
&\left(1-\beta_{F_A}^3(x_j)\right)\left(1-\beta_{F_B}^3(x_j)\right)+\pi_{F_A}^3(x_j)\pi_{F_B}^3(x_j)],\\
&GT_{FFS}(F_{A_j}) = [(1-\alpha_{F_A}^3(x_j))^2+(1-\beta_{F_A}^3(x_j))^2+\pi_{F_A}^6(x_j)],\\
&GT_{FFS}(F_{B_j}) = [(1-\alpha_{F_B}^3(x_j))^2+(1-\beta_{F_B}^3(x_j))^2+\pi_{F_B}^6(x_j)],\\
&0 \leq \alpha_{F_A}(x_j) \leq 1,\ 0 \leq \beta_{F_A}(x_j) \leq 1,\\
&0 \leq \alpha_{F_A}^3(x_j)+\beta_{F_A}^3(x_j) \leq 1,\\
&\pi_{F_A}(x_j) = \sqrt[3]{1-\alpha_{F_A}^3(x_j)-\beta_{F_A}^3(x_j)},\\
&0 \leq \alpha_{F_B}(x_j) \leq 1,\ 0 \leq \beta_{F_B}(x_j) \leq 1,\\
&0 \leq \alpha_{F_B}^3(x_j)+\beta_{F_B}^3(x_j) \leq 1,\\
&\pi_{F_B}(x_j) = \sqrt[3]{1-\alpha_{F_B}^3(x_j)-\beta_{F_B}^3(x_j)},\ 0 \leq \rho \leq 1 \text{ and } 1 \leq j \leq n.
\end{aligned}
$$

The above $IGD_{FFS}(F_A, F_B)$ also satisfies the following properties:

(1) $0 \leq IGD_{FFS}(F_A, F_B) \leq 1$;

(2) $IGD_{FFS}(F_A, F_B) = IGD_{FFS}(F_B, F_A)$;

(3) $IGD_{FFS}(F_A, F_A) = 1$;

where $F_A = \{<x_j, \alpha_{F_A}(x_j), \beta_{F_A}(x_j)> \mid x_j \in X\}$, $F_B = \{<x_j, \alpha_{F_B}(x_j), \beta_{F_B}(x_j)> \mid x_j \in X\}$ and $X = \{x_1, x_2, \ldots, x_n\}$.

The proofs of the three properties of $IGD_{FFS}(F_A, F_B)$ are similar to the ones of $ID_{FFS}(F_A, F_B)$.

Definition 3.5: The improved WGDSM between two FFSs F_A and F_B in the universe of discourse X, where $X = \{x_1, x_2, \ldots, x_n\}$, $F_A = \{<x_j, \alpha_{F_A}(x_j), \beta_{F_A}(x_j)> \mid x_j \in X\}$ and $F_B = \{<x_j, \alpha_{F_B}(x_j), \beta_{F_B}(x_j)> \mid x_j \in X\}$ is defined as follows:

$$
IWGD_{FFS}(F_A, F_B) = \sum_{j=1}^n \omega_j \frac{WGC_{FFS}(F_{A_j}, F_{B_j})}{\rho WGT_{FFS}(F_A)+(1-\rho)WGT_{FFS}(F_B)}, \tag{15}
$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector, $\sum_{j=1}^{n}\omega_{j=1}$,

$$WGC_{FFS}(F_{A_j}, F_{B_j}) = [(1-\alpha^3_{F_A}(x_j))(1-\alpha^3_{F_B}(x_j)) + (1-\beta^3_{F_A}(x_j))(1-\beta^3_{F_B}(x_j)) + \pi^3_{F_A}(x_j)\pi^3_{F_B}(x_j)],$$

$$WGT_{FFS}(F_{A_j}) = [(1-\alpha^3_{F_A}(x_j))^2 + (1-\beta^3_{F_A}(x_j))^2 + \pi^6_{F_A}(x_j)],$$

$$WGT_{FFS}(F_{B_j}) = [(1-\alpha^3_{F_B}(x_j))^2 + (1-\beta^3_{F_B}(x_j))^2 + \pi^6_{F_B}(x_j)],$$

$$0 \le \alpha_{F_A}(x_j) \le 1,\ 0 \le \beta_{F_A}(x_j) \le 1,$$

$$0 \le \alpha^3_{F_A}(x_j) + \beta^3_{F_A}(x_j) \le 1,$$

$$\pi_{F_A}(x_j) = \sqrt[3]{1-\alpha^3_{F_A}(x_j) - \beta^3_{F_A}(x_j)},$$

$$0 \le \alpha_{F_B}(x_j) \le 1,\ 0 \le \beta_{F_B}(x_j) \le 1,$$

$$0 \le \alpha^3_{F_B}(x_j) + \beta^3_{F_B}(x_j) \le 1,$$

$$\pi_{F_B}(x_j) = \sqrt[3]{1-\alpha^3_{F_B}(x_j) - \beta^3_{F_B}(x_j)},\ 0 \le \rho \le 1 \text{ and } 1 \le j \le n.$$

The above $IWGD_{FFS}(F_A, F_B)$ also satisfies the following properties:

(1) $0 \le IWGD_{FFS}(F_A, F_B) \le 1$;

(2) $IWGD_{FFS}(F_A, F_B) = IWGD_{FFS}(F_B, F_A)$;

(3) $IWGD_{FFS}(F_A, F_A) = 1$;

where $F_A = \{<x_j, \alpha_{F_A}(x_j), \beta_{F_A}(x_j)> \mid x_j \in X\}$, $F_B = \{<x_j, \alpha_{F_B}(x_j), \beta_{F_B}(x_j)> \mid x_j \in X\}$ and $X = \{x_1, x_2, \ldots, x_n\}$.

The proofs of the three properties of $IWGD_{FFS}(F_A, F_B)$ are similar to the ones of $IWD_{FFS}(F_A, F_B)$.

4 A New MCDM Method Based on the Porposed Improved WDSM/WGDSM and Numerical Examples

In this section, we propose a new MCDM method and use two examples to compare the proposed method with Lee *et al.*'s method [14] for DM under FF environments shown as below.

4.1 The Proposed MCDM Method

At present, we propose a new MCDM method based on the proposed improved WDSM or WGDSM to solve DM problems under FF environments. Assuming that $A = \{A_1, A_2, \ldots, A_m\}$ is a set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ is a set of criteria. Let K_b be a collection of benefit criteria and let K_c be a collection of cost criteria, where $K_b \cap K_c = \phi$. Let $D = (d_{ij})_{m\times n} = ((x_{ij}, y_{ij}, z_{ij}))_{m\times n}$ be the FFN decision matrix given by the decision-maker, where x_{ij} is the degree of membership, y_{ij} is the degree of non-membership and $z_{ij} = \sqrt[3]{1-x_{ij}^3 - y_{ij}^3}$ is the degree of indeterminacy, $1 \le i \le m$ and $1 \le j \le n$. The steps of the proposed MCDM method is shown as follows:

Step 1: Construct the FFN decision matrix $D = (d_{ij})_{m\times n} = ((x_{ij}, y_{ij}))_{m\times n}$ shown as follows:

$$D = (d_{ij})_{m\times n} = ((x_{ij}, y_{ij}))_{m\times n} = \begin{array}{c|cccc} & C_1 & C_2 & \cdots & Cn \\ \hline A_1 & (x_{11}, y_{11}) & (x_{12}, y_{12}) & \cdots & (x_{1n}, y_{1n}) \\ A_2 & (x_{21}, y_{21}) & (x_{22}, y_{22}) & \cdots & (x_{2n}, y_{2n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_m & (x_{m1}, y_{m1}) & (x_{m2}, y_{m2}) & \cdots & (x_{mn}, y_{mn}) \end{array},$$

where $0 \le x_{ij}^3 + y_{ij}^3 \le 1$, $z_{ij} = \sqrt[3]{1-x_{ij}^3 - y_{ij}^3}$, $1 \le i \le m$ and $1 \le j \le n$.

Step 2: The weight vector for each criterion is denoted by $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ which can be given by the decision-maker. If the weights of criteria are not given by the decision-maker subjectively, we can calculate the weights of criteria in objective way. Through the following substeps, we can get the weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ for the criteria $C_1, C_2, \ldots$, and C_n shown as below:

Step 2.1: Calculate the entropy measure of every FFN [5] of decision matrix shown as follows:

$$E_{ij} = 1 - \left[(x_{ij}^3 - y_{ij}^3)(x_{ij}^3 + y_{ij}^3)\right]^2, \tag{16}$$

where $1 \le i \le m$ and $1 \le j \le n$. Then, the decision matrix D is transformed into an entropy measure matrix (EMM):

$$D_{EM} = (E_{ij})_{m\times n} = \begin{array}{c|cccc} & C_1 & C_2 & \cdots & Cn \\ \hline A_1 & (E_{11}) & (E_{12}) & \cdots & (E_{1n}) \\ A_2 & (E_{21}) & (E_{22}) & \cdots & (E_{2n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_m & (E_{m1}) & (E_{m2}) & \cdots & (E_{mn}) \end{array}.$$

Step 2.2: Normalize the entropy values of the EMM using the following equations:

$$N_{i1} = \frac{E_{i1}}{MAX(E_{i1})}, N_{i2} = \frac{E_{i2}}{MAX(E_{i2})}, \ldots, N_{in} = \frac{E_{in}}{MAX(E_{in})}, \tag{17}$$

where $1 \le i \le m$. Then the entropy measure matrix D_{EM} is transformed into the normalized matrix (NM):

$$D_N = (N_{ij})_{m\times n} = \begin{array}{c|cccc} & C_1 & C_2 & \cdots & Cn \\ \hline A_1 & (N_{11}) & (N_{12}) & \cdots & (N_{1n}) \\ A_2 & (N_{21}) & (N_{22}) & \cdots & (N_{2n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_m & (N_{m1}) & (N_{m2}) & \cdots & (N_{mn}) \end{array}.$$

Step 2.3: Calculate the objective weight ω_k for criterion C_k shown as follows:

$$\omega_k = \frac{1-\sum_{i=1}^{m} N_{ik}}{n - \sum_{j=1}^{n}\left(\sum_{i=1}^{m} N_{ij}\right)}, \tag{18}$$

where $1 \le k \le n$.

Step 3: Define the Fermatean fuzzy positive ideal solution (FFPIS) F^+ shown as follows:

$$F^+ = \left\{f_1^+, f_2^+, ..., f_n^+\right\} = \left\{(x_1^+, y_1^+, z_1^+), (x_2^+, y_2^+, z_2^+), ..., (x_n^+, y_n^+, z_n^+)\right\}, \quad \textbf{(19)}$$

where if $C_j \in K_b$, then let $f_j^+ = (x_j^+, y_j^+) = (1, 0)$; if $C_j \in K_c$, then let $f_j^+ = (x_j^+, y_j^+) = (0, 1)$, where $z_j^+ = \sqrt[3]{1-(x_j^+)^3-(y_j^+)^3}$ and $1 \le j \le n$.

Step 4: Calculate the improved WDSM or WGDSM between the elements at the ith row of the FFN decision matrix $D = (d_{ij})_{m\times n}$ and the elements in the obtained FFPIS F^+ based on Eq. (13) or Eq. (15) with the weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ for the criteria C_1, C_2, …, and C_n obtained from **Step 2** shown as follows:

$$IWD_{FFS}(F_{A_i}, F^+) = \sum_{j=1}^{n} \omega_j \frac{2WC_{FFS}(F_{A_i}, F^+)}{WT_{FFS}(F_{A_i}) + WT_{FFS}(F^+)},$$

where

$$F_{A_i} = \{(x_{i1}, y_{i1}), (x_{i2}, y_{i2}), ..., (x_{in}, y_{in}) | 1 \le i \le m\},$$
$$WC_{FFS}(F_{A_i}, F^+) = [(1-x_{ij}^3)(1-(x_j^+)^3) + (1-y_{ij}^3)(1-(y_j^+)^3) + z_{ij}^3(z_j^+)^3],$$
$$WT_{FFS}(F_{A_i}) = [(1-x_{ij}^3)^2 + (1-y_{ij}^3)^2 + z_{ij}^6],$$
$$WT_{FFS}(F^+) = [(1-(x_j^+)^3)^2 + (1-(y_j^+)^3)^2 + (z_j^+)^6],$$

or

$$IWGD_{FFS}(F_{A_i}, F^+) = \sum_{j=1}^{n} \omega_j \frac{WGC_{FFS}(F_{A_i}, F^+)}{\rho WGT_{FFS}(F_{A_i}) + (1-\rho) WGT_{FFS}(F^+)},$$

where

$$F_{A_i} = \{(x_{i1}, y_{i1}), (x_{i2}, y_{i2}), ..., (x_{in}, y_{in}) | 1 \le i \le m\},$$
$$WGC_{FFS}(F_{A_i}, F^+) = [(1-x_{ij}^3)(1-(x_j^+)^3) + (1-y_{ij}^3)(1-(y_j^+)^3) + z_{ij}^3(z_j^+)^3],$$
$$WGT_{FFS}(F_{A_i}) = [(1-x_{ij}^3)^2 + (1-y_{ij}^3)^2 + z_{ij}^6],$$
$$WGT_{FFS}(F^+) = [(1-(x_j^+)^3)^2 + (1-(y_j^+)^3)^2 + (z_j^+)^6],$$

The larger the value of $IWD_{FFS}(F_{A_i}, F^+)$ or $IWGD_{FFS}(F_{A_i}, F^+)$, the better the preference order of alternative A_i, where $1 \le i \le m$.

4.2 Illustrative Examples

Example 4.1 **[14]**: Suppose that a company want to choose the right sustainable and green building materials supplier from five alternatives A_1, A_2, A_3, A_4 and A_5 for their future development. There are four criteria in the assessment shown as follows:

C_1: Business credit,
C_2: Technical capability,
C_3: Quality level, and
C_4: Price,

where the criteria C_1, C_2 and C_3 are benefit criteria and the criterion C_4 is cost criterion. If the IFN decision matrix $D = (d_{ij})_{5\times4}$ is given by the decision-maker. It is clear that an IFN decision matrix is also an FFN decision matrix.

[Step 1]: Construct the FFN decision matrix $D = (d_{ij})_{5\times4}$ shown as follows:

$$D = (d_{ij})_{5\times4} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} (0.712, 0.157) & (0.491, 0.263) & (0.627, 0.183) & (0.635, 0.217) \\ (0.628, 0.239) & (0.562, 0.197) & (0.582, 0.195) & (0.619, 0.205) \\ (0.537, 0.296) & (0.612, 0.189) & (0.631, 0.209) & (0.597, 0.196) \\ (0.691, 0.162) & (0.582, 0.201) & (0.609, 0.253) & (0.681, 0.192) \\ (0.586, 0.177) & (0.627, 0.125) & (0.573, 0.181) & (0.592, 0.182) \end{bmatrix} \end{array}.$$

[Step 2]: Calculate the objective weights of criteria through the following substeps:

[Step 2.1]: Based Eq. (16), calculate the entropy measure of every element of decision matrix D shown as follows:

$E_{11} = 0.9830$, $E_{12} = 0.9998$, $E_{13} = 0.9963$, $E_{14} = 0.9957$,
$E_{21} = 0.9963$, $E_{22} = 0.9990$, $E_{23} = 0.9985$, $E_{24} = 0.9968$,
$E_{31} = 0.9995$, $E_{32} = 0.9972$, $E_{33} = 0.9960$, $E_{34} = 0.9980$,
$E_{41} = 0.9882$, $E_{42} = 0.9985$, $E_{43} = 0.9974$, $E_{44} = 0.9901$,
$E_{51} = 0.9984$, $E_{52} = 0.9963$, $E_{53} = 0.9987$, $E_{54} = 0.9982$.

The EMM is shown as below.

$$D_{EM} = (E_{ij})_{5\times4} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 0.9830 & 0.9998 & 0.9963 & 0.9957 \\ 0.9963 & 0.9990 & 0.9985 & 0.9968 \\ 0.9995 & 0.9972 & 0.9960 & 0.9980 \\ 0.9882 & 0.9985 & 0.9974 & 0.9901 \\ 0.9984 & 0.9963 & 0.9987 & 0.9982 \end{bmatrix} \end{array}.$$

[Step 2.2]: Use the equations in Eq. (17) to normalize the entropy values of the EMM to get the NM:

$$D_N = (N_{ij})_{5\times4} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 0.9836 & 1.0000 & 0.9976 & 0.9976 \\ 0.9968 & 0.9992 & 0.9997 & 0.9987 \\ 1.0000 & 0.9974 & 0.9973 & 0.9998 \\ 0.9887 & 0.9987 & 0.9987 & 0.9919 \\ 0.9989 & 0.9965 & 1.0000 & 1.0000 \end{bmatrix} \end{array}.$$

[Step 2.3]: Based on Eq. (18), calculate the objective weight ω_k for criterion C_k, where $1 \leq k \leq 4$ shown as follows:

$$\omega_1 = 0.2489,\ \omega_2 = 0.2504,\ \omega_3 = 0.2505,\ \omega_4 = 0.2502.$$

[Step 3]: Based on Eq. (19), define the FFPIS F^+ shown as follows:

$$F^+ = \{(1, 0), (1, 0), (1, 0), (0, 1)\},$$

where the criteria C_1, C_2 and C_3 are benefit criteria and the criterion C_4 is cost criterion.

[Step 4]: Based on Eq. (15) and the weight vector $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)^T = (0.2489, 0.2504, 0.2505, 0.2502)^T$ for the criteria C_1, C_2, C_3 and C_4 obtained from **[Step 2]**, calculate the improved WGDSM between the elements at the ith row of the FFN decision matrix $D = (d_{ij})_{5\times4}$ and the elements in the obtained FFPIS F^+, where $1 \leq i \leq 5$ and $1 \leq j \leq 4$ shown as follows:

$$IWGD_{FFS}(F_{A_1}, F^+) = \sum_{j=1}^{4} \omega_j \frac{2WGC_{FFS}(F_{A_1}, F^+)}{\rho WGT_{FFS}(F_{A_1}) + (1-\rho) WGT_{FFS}(F^+)},$$

$$IWGD_{FFS}(F_{A_2}, F^+) = \sum_{j=1}^{4} \omega_j \frac{2WGC_{FFS}(F_{A_2}, F^+)}{\rho WGT_{FFS}(F_{A_2}) + (1-\rho) WGT_{FFS}(F^+)},$$

$$IWGD_{FFS}(F_{A_3}, F^+) = \sum_{j=1}^{4} \omega_j \frac{2WGC_{FFS}(F_{A_3}, F^+)}{\rho WGT_{FFS}(F_{A_3}) + (1-\rho) WGT_{FFS}(F^+)},$$

$$IWGD_{FFS}(F_{A_4}, F^+) = \sum_{j=1}^{4} \omega_j \frac{2WGC_{FFS}(F_{A_4}, F^+)}{\rho WGT_{FFS}(F_{A_4}) + (1-\rho) WGT_{FFS}(F^+)},$$

$$IWGD_{FFS}(F_{A_5}, F^+) = \sum_{j=1}^{4} \omega_j \frac{2WGC_{FFS}(F_{A_5}, F^+)}{\rho WGT_{FFS}(F_{A_5}) + (1-\rho) WGT_{FFS}(F^+)},$$

where

$F_{A_1} = \{(0.712, 0.157), (0.491, 0.263), (0.627, 0.183), (0.635, 0.217)\}$,
$F_{A_2} = \{(0.628, 0.239), (0.562, 0.197), (0.582, 0.195), (0.619, 0.205)\}$,
$F_{A_3} = \{(0.537, 0.296), (0.612, 0.189), (0.631, 0.209), (0.597, 0.196)\}$,
$F_{A_4} = \{(0.691, 0.162), (0.582, 0.201), (0.609, 0.253), (0.681, 0.192)\}$ and
$F_{A_5} = \{(0.586, 0.177), (0.627, 0.125), (0.573, 0.181), (0.592, 0.182)\}$.

When $\rho = 0.1$, we can get $IWGD_{FFS}(F_{A_1}, F^+) = 0.8356$, $IWGD_{FFS}(F_{A_2}, F^+) = 0.8331$, $IWGD_{FFS}(F_{A_3}, F^+) = 0.8362$, $IWGD_{FFS}(F_{A_4}, F^+) = 0.8264$ and $IWGD_{FFS}(F_{A_5}, F^+) = 0.8414$. Because $IWGD_{FFS}(F_{A_5}, F^+) > IWGD_{FFS}(F_{A_3}, F^+) > IWGD_{FFS}(F_{A_1}, F^+) > IWGD_{FFS}(F_{A_2}, F^+) > IWGD_{FFS}(F_{A_4}, F^+)$, the preference order of the alternatives A_1, A_2, A_3, A_4 and A_5 is: $A_5 > A_3 > A_1 > A_2 > A_4$. The best alternative, i.e., A_5, obtained by the proposed method is coincided with Lee *et al.*'s method [14]. The advantage of the proposed method is that it is more powerful and flexible than Lee *et al.*'s method [14] for MCDM under IF/FF environments. However, the major drawback of using Lee *et al.*'s method [14] is that it cannot get the preference order of the alternatives under some FF environments, explaining in ***Example 4.2***.

Example 4.2: Assuming that there are five alternatives: A_1, A_2, A_3, A_4 and A_5 and assuming that there are four criteria: C_1, C_2, C_3 and C_4, where the criteria C_1, C_2 and C_3 are benefit criteria and the criterion C_4 is cost criterion. If the FFN decision matrix $D = (d_{ij})_{5\times4}$ is given by the decision-maker.

[Step 1]: Construct the FFN decision matrix $D = (d_{ij})_{5\times4}$ shown as follows:

$$D = (d_{ij})_{5\times4}$$

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} (0.750, 0.280) & (0.491, 0.263) & (0.627, 0.183) & (0.635, 0.217) \\ (0.628, 0.239) & (0.660, 0.370) & (0.582, 0.195) & (0.619, 0.205) \\ (0.537, 0.296) & (0.612, 0.189) & (0.780, 0.380) & (0.597, 0.196) \\ (0.691, 0.162) & (0.582, 0.201) & (0.609, 0.253) & (0.880, 0.310) \\ (0.586, 0.177) & (0.627, 0.125) & (0.573, 0.181) & (0.592, 0.182) \end{bmatrix} \end{array}.$$

[Step 2]: Calculate the objective weights of criteria through the following substeps:

[Step 2.1]: Based Eq. (16), calculate the entropy measure of every element of decision matrix D and the EMM is shown as below.

$$D_{EM} = (E_{ij})_{5\times4} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 0.9685 & 0.9998 & 0.9963 & 0.9957 \\ 0.9963 & 0.9936 & 0.9985 & 0.9968 \\ 0.9995 & 0.9972 & 0.9506 & 0.9980 \\ 0.9882 & 0.9985 & 0.9974 & 0.7852 \\ 0.9984 & 0.9963 & 0.9987 & 0.9982 \end{bmatrix} \end{array}.$$

[Step 2.2]: Use the equations in Eq. (17) to normalize the entropy values of the EMM to get the NM:

$$D_N = (N_{ij})_{5\times4} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 0.9690 & 1.0000 & 0.9976 & 0.9976 \\ 0.9968 & 0.9938 & 0.9997 & 0.9987 \\ 1.0000 & 0.9974 & 0.9518 & 0.9998 \\ 0.9887 & 0.9987 & 0.9987 & 0.7866 \\ 0.9989 & 0.9965 & 1.0000 & 1.0000 \end{bmatrix} \end{array}.$$

[Step 2.3]: Based on Eq. (18), calculate the objective weight ω_k for criterion C_k, where $1 \leq k \leq 4$ shown as follows:

$$\omega_1 = 0.2523,\ \omega_2 = 0.2544,\ \omega_3 = 0.2519,\ \omega_4 = 0.2414.$$

[Step 3]: Based on Eq. (19), define the FFPIS F^+ shown as follows:

$$F^+ = \{(1, 0), (1, 0), (1, 0), (0, 1)\},$$

where the criteria C_1, C_2 and C_3 are benefit criteria and the criterion C_4 is cost criterion.

[Step 4]: Based on Eq. (15) and the weight vector $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)^T = (0.2523, 0.2544, 0.2519, 0.2414)^T$ for the criteria C_1, C_2, C_3 and C_4 obtained from **[Step 2]**, calculate the improved WGDSM between the elements at the *i*th row of the FFN decision matrix $D = (d_{ij})_{5\times 4}$ and the elements in the obtained FFPIS F^+, where $1 \le i \le 5$ and $1 \le j \le 4$ shown as follows:

$$IWGD_{FFS}(F_{A_1}, F^+) = \sum_{j=1}^{4} \omega_j \frac{2WGC_{FFS}(F_{A_1}, F^+)}{\rho WGT_{FFS}(F_{A_1}) + (1-\rho)WGT_{FFS}(F^+)},$$

$$IWGD_{FFS}(F_{A_2}, F^+) = \sum_{j=1}^{4} \omega_j \frac{2WGC_{FFS}(F_{A_2}, F^+)}{\rho WGT_{FFS}(F_{A_2}) + (1-\rho)WGT_{FFS}(F^+)},$$

$$IWGD_{FFS}(F_{A_3}, F^+) = \sum_{j=1}^{4} \omega_j \frac{2WGC_{FFS}(F_{A_3}, F^+)}{\rho WGT_{FFS}(F_{A_3}) + (1-\rho)WGT_{FFS}(F^+)},$$

$$IWGD_{FFS}(F_{A_4}, F^+) = \sum_{j=1}^{4} \omega_j \frac{2WGC_{FFS}(F_{A_4}, F^+)}{\rho WGT_{FFS}(F_{A_4}) + (1-\rho)WGT_{FFS}(F^+)},$$

$$IWGD_{FFS}(F_{A_5}, F^+) = \sum_{j=1}^{4} \omega_j \frac{2WGC_{FFS}(F_{A_5}, F^+)}{\rho WGT_{FFS}(F_{A_5}) + (1-\rho)WGT_{FFS}(F^+)},$$

where

F_{A_1} = {(0.750, 0.280), (0.491, 0.263), (0.627, 0.183), (0.635, 0.217)},
F_{A_2}= {(0.628, 0.239), (0.660, 0.370), (0.582, 0.195), (0.619, 0.205)},
F_{A_3}= {(0.537, 0.296), (0.612, 0.189), (0.780, 0.380), (0.597, 0.196)},
F_{A_4}= {(0.691, 0.162), (0.582, 0.201), (0.609, 0.253), (0.880, 0.310)} and
F_{A_5}= {(0.586, 0.177), (0.627, 0.125), (0.573, 0.181), (0.592, 0.182)}.

The values of the improved WGDSM and preference orders with different ρ values are depicted in Table 1.

We can see that $x_{11}^3 + y_{11}^3 = 0.4438 \le 1$, $x_{22}^3 + y_{22}^3 = 0.3381 \le 1$, $x_{33}^3 + y_{33}^3 = 0.5294 \le 1$ and $x_{44}^3 + y_{44}^3 = 0.7113 \le 1$, where (x_{11}, y_{11}) = (0.750, 0.280), (x_{22}, y_{22}) = (0.660, 0.370), (x_{33}, y_{33}) = (0.780, 0.380) and (x_{44}, y_{44}) = (0.880, 0.310), thus they are all FFNs. Since $x_{11} + y_{11} = 1.030 \nleqslant 1$, $x_{22} + y_{22} = 1.030 \nleqslant 1$, $x_{33} + y_{33} = 1.160 \nleqslant 1$ and $x_{44} + y_{44} = 1.190 \nleqslant 1$, where (x_{11}, y_{11}) = (0.750, 0.280), (x_{22}, y_{22}) = (0.660, 0.370), (x_{33}, y_{33}) = (0.780, 0.380) and (x_{44}, y_{44}) = (0.880, 0.310), thus the four elements of the matrix D: (x_{11}, y_{11}), (x_{22}, y_{22}), (x_{33}, y_{33}) and (x_{44}, y_{44}) cannot be considered as IFNs. Therefore, the drawback and limitations of Lee *et al.*'s method [14] is that it cannot deal with MCDM problems under some FF environments. That is, Lee *et al.*'s method [14] cannot get the preference order of the alternatives of ***Example 4.2***.

5 Conclusion

In this paper, we have proposed a new MCDM method based on FFSs and improved DSM/ GDSM between two FFSs with completely unknown weights of criteria. We have used two examples to compare the experimental results of the proposed method with Lee *et al.*'s method [14]. Our experimental results show that the proposed method is more powerful and flexible than Lee *et al.*'s method [14] for MCDM under IF/FF environments and can overcome the drawbacks and limitations of some existing methods that they cannot get the preference order of the alternatives under FF environments. Therefore, the proposed method provides us an effective way to solve MCDM problems under IF/ FF environments. We will strive to develop FF weighted averaging and weighted geometric aggregation operators that can be used for MCDM problems in the very near future. It is worth further research to extend the proposed method to develop MCDM methods and multiple criteria group DM methods for more uncertain problems under interval-valued Fermatean fuzzy [17] environments.

Table 1. The values of the improved WGDSM and preference orders with different ρ values

ρ	$IWGD_{FFS}(F_{A_1}, F^+)$	$IWGD_{FFS}(F_{A_2}, F^+)$	$IWGD_{FFS}(F_{A_3}, F^+)$	$IWGD_{FFS}(F_{A_4}, F^+)$	$IWGD_{FFS}(F_{A_5}, F^+)$	Preference order
0.1	0.8377	0.8346	0.8418	0.7533	0.8430	$A_5 > A_3 > A_1 > A_2 > A_4$
0.3	0.7054	0.6991	0.7152	0.6404	0.6918	$A_3 > A_1 > A_2 > A_5 > A_4$
0.5	0.6108	0.6018	0.6246	0.5591	0.5866	$A_3 > A_1 > A_2 > A_5 > A_4$
0.7	0.5394	0.5285	0.5560	0.4977	0.5092	$A_3 > A_1 > A_2 > A_5 > A_4$
0.9	0.4835	0.4712	0.5021	0.4495	0.4499	$A_3 > A_1 > A_2 > A_5 > A_4$

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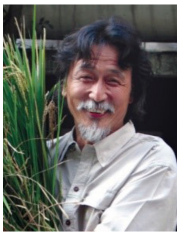

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