Fault Tolerance of Bubble-sort Networks on Components

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Abstract

Reliability evaluation of interconnection networks are important to the design and maintenance of multiprocessor systems. The component connectivity is an important measure for the reliability of interconnection networks.

The bubble-sort network B_n is a popular underlying topology for distributed systems. The t-component (edge) connectivity $\kappa^t(G)$ ($\lambda^t(G)$) of a graph $G = (V, E)$ is the minimum vertex (edge) number of a set $F \subset V$ such that $G - F$ is not connected and $G - F$ has at least t components. In this paper, we determine the $\kappa^t(B_n)$ and $\lambda^t(B_n)$ for small t.

Keywords: Distributed systems, Interconnection networks, Bubble-sort graphs, Fault tolerance

1 Introduction

Interconnection networks are the underlying topologies for many large-scale multi- processor systems. A multiprocessor system may contain hundreds or even thousands of nodes, some of which may be faulty when the system is implemented. As the number of processors in a system increases, the possibility that its processors may become faulty also increases. Because designing such systems without defects is nearly impossible, reliability and fault tolerance are two of the most critical concerns of multiprocessor systems.

A graph $G = (V, E)$ with n vertices and m edges consists of the vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set $E = \{e_1, e_2, ..., e_m\}$, where each edge consists of two vertices called, endpoints. An element in V is called a vertex of G. An element of E is called an edge of G. We may also use a node for a vertex and a link for an edge in this paper. A graph $G = (V, E)$ can represent the interconnection network, where every node in V denotes a processor, and every edge in E denotes a communication link. Failures of processors and links are inevitable in a large-scale multiprocessor system. The connectivity of graphs can measure the fault tolerance of networks. The possibility of failure with all neighbors of a vertex or all edges incident to a vertex is very small. Hence there are many kinds of connectivities to assess the fault tolerance of interconnection networks. Fault tolerance of interconnection networks becomes an essential problem and has been widely studied, such as, [1] hamiltonian path of Bubble-sort graphs with edge faults, [6, 8] distance connectivity of graphs, [10] pessimistic diagnosability of Cayley graphs generated by transpositions, [12] extra connectivity of bubble-sort star graphs, [15] g-extra conditional diagnosability of hierarchical cubic networks, [20] conditional connectivity of Cayley graphs generated by unicyclic graphs.

The classic parameter is the edge connectivity $\lambda(G)$ and connectivity $\kappa(G)$. In general, the larger $\lambda(G)$ or $\kappa(G)$ is, the more stable the network is. Notice that $\kappa(G) \leq \lambda(G) \leq \delta(G)$. In general, the larger $\lambda(G)$ or $\kappa(G)$ is, the more stable the network is. Notice that $\kappa(G) \leq$ If every minimum vertex cut $F = N(u)$ or minimum edge cut $F = E(u)$ for some vertex u, then graph G is super-κ or super-λ. If G is super-κ or super-λ, then we obtain that $\kappa(G) = \delta(G)$ or $\lambda(G) = \delta(G)$. If the graph $G - S$ is not connected and each component of $G - S$ has more than one vertex, then the edge set or vertex set $S \subset E$ is said to be a restricted edge cut or a restricted cut. The restricted edge connectivity $\lambda'(G)$ or restricted connectivity $\kappa'(G)$ of G is the edge number of a minimum restricted edge cut or the vertex number of a minimum restricted cut of G. Let T be a vertex set of G. If $G - T$ has at least t components, then T is a t-component cut of G. The t-component connectivity $\kappa^t(G)$ of G is the vertex number of a smallest t-component cut. We can obtain the t-component edge connectivity $\lambda^t(G)$ similarly. The t-component (edge) connectivity was introduced in [3] and [16] independently. It can be seen that $\kappa^{t+1}(G) \geq \kappa^t(G)$ and $\lambda^{t+1}(G) \geq \lambda^t(G)$ for each integer $t \geq 1$. By the definition, we have that $\kappa(G) = \kappa^2(G)$ and $\lambda(G) = \lambda^2(G)$. For more references about component connectivity, we can see [4] locally twisted cubes, [5] twisted cubes, [7] BC networks, [9] regular networks, [11] alternating group graphs and split-star networks, [13, 21] hypercubes, and so on.

Definition 1. The vertex set V of bubble-sort network B_n, $V(B_n) = \{x_1x_2 ... x_n : x_1x_2 ... x_n$ is a permutation on $\{1, 2, ..., n\}\}$. Two distinct vertices $y = y_1y_2 ... y_n$ and x

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$= x_1x_2 \ldots x_n$ in B_n have an edge if and only if y is got by swapping the two elements x_i and x_{i+1} of x, i.e., $y_{i+1} = x_i$ and $y_i = x_{i+1}$. We can see Figure 1.

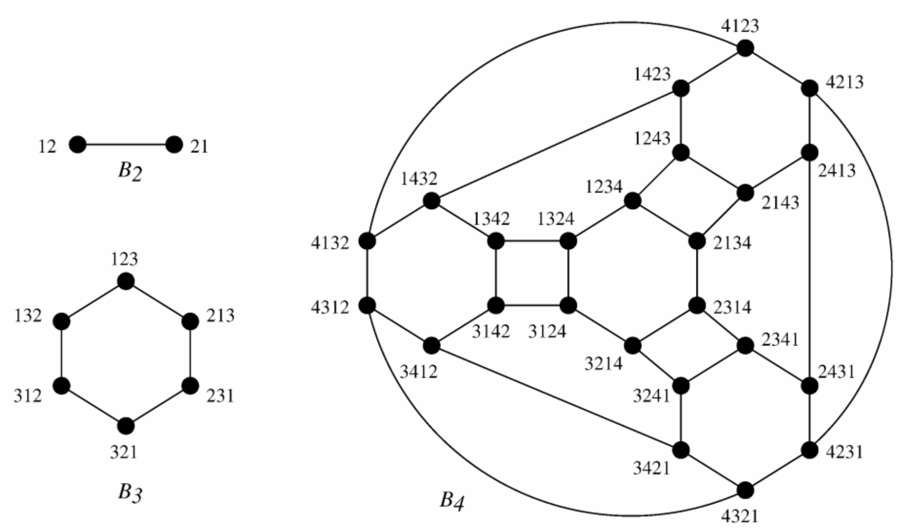

Figure 1. Illustration of the bubble-sort graphs B_2, B_3, B_4

The bubble-sort network has n! vertices and is $n - 1$ regular, bipartite, vertex transitive. It can be defined recursively. The bubble-sort graphs, which belong to the class of Cayley graphs, have been attractive alternative to the hypercubes. They have some good topological properties such as highly symmetry and recursive structure. The bubble-sort network B_n can be partitioned into n subgraphs $B_n(i)$, where $B_n(i)$ is induced by the vertex set $\{x_1x_2 \ldots x_n : x_n = i\}$ $(1 \leq i \leq n)$ and $B_n(i)$ is isomorphic to B_{n-1}. And the number of edges between $B_n(i)$ and $B_n(j)$ is $(n - 2)!$. Many authors study the properties of bubble-sort graphs, for example, [14] generalized 3-connectivity, [17] faulty nodes, [18] decycling number, [19] connectivity and super connectivity.

Let $G = (V, E)$ be a connected graph. We use $N(v)$ to denote the neighborhood of a vertex v and $E(v)$ to denote the edges incident to v. Suppose $T \subset V$, $G[T]$ is the subgraph induced by T, $N_G(T) = \cup_{v \in T} N(v) - T$, $N_G[T] = N_G(T) \cup T$, and $G - T$ represents the subgraph of G induced by the set of $V - T$. For X, $Y \subset V$, $[X, Y]$ represents the edge set of G in which one end is in X and the other is in Y. For all the notations and symbols, we can see Table 1.

Table 1. Notations and symbols

G	a graph
V, V (G)	vertex set of a graph
E, E(G)	edge set of a graph
$\lambda(G)$, $\kappa(G)$	connectivity of a graph
$\lambda'(G)$, $\kappa'(G)$	restricted connectivity of a graph
$\lambda^t(G)$, $\kappa^t(G)$	component connectivity of a graph
$\delta(G)$	minimum degree of a graph
B_n	the bubble-sort network
N (v)	the neighborhood of a vertex v
E(v)	the edges incident to v
G[T]	the subgraph induced by T
$G - T$	the subgraph of G induced by the set of $V - T$
$N_G(T)$	$\cup_{v \in T} N(v) - T$
$N_G[T]$	$N_G(T) \cup T$
[X, Y]	the edge set of G in which one end is in X and the other is in Y

Throughout this paper, we only consider undirected simple connected graphs. We follow Bondy and Murty [2] for terminology and definitions.

Our aim in this paper is to determine the $\kappa^t(B_n)$ and $\lambda^t(B_n)$ for small t. In the last, we give a conclusion.

2 Component Connectivity of Bubble-sort Networks

In [19], the authors determined the connectivity and restricted connectivity of bubble-sort networks.

Lemma 2.1. [19] $\kappa(B_n) = \lambda(B_n) = n - 1 (n \geq 2)$.
Lemma 2.2. [19] $\kappa'(B_n) = \lambda'(B_n) = 2n - 4 (n \geq 3)$.

From Lemma 2.1 and 2.2, we can derive Proposition 2.3 and Theorem 2.4 easily.

Proposition 2.3. B_n is super-λ and super-κ $(n \geq 4)$.

Theorem 2.4. $\lambda^{2\prime}$ $(B_n) = n-1$ $(n \geq 2)$.

We will obtain the component connectivity of B_n.

Theorem 2.5. $\lambda^3(B_n) = 2n - 3 (n \geq 3)$.

Proof. We will prove it by induction. if $n = 3$, then it is true. We assume that $n \geq 4$ and the result holds when $n \leq k - 1$. We will prove it for $n = k$.

Take an edge $e = xy$ and $F = E(x) \cup E(y)$. Then $|F| = 2n - 3$ and $B_n - F$ has at least three components. Hence $\lambda^3(B_n) \leq 2n - 3$. It suffices to show $\lambda^3(B_n) \geq 2n - 3$.

Suppose that there is an edge set F with $|F| \leq 2n - 4$, and $B_n - F$ has at least three components. Since B_n can be decomposed to n subgraphs $B_n(i)(i = 1, 2, ..., n)$, we set $F_i = E(B_n(i)) \cap F$ $(i = 1, 2, ..., n)$. Without loss of generality, we assume$|F_1| \geq |F_2| \geq ... \geq |F_n|$. Note that $\lambda(B_{n-1}) = n - 2$.

Case 1. $|F_1| \leq n - 3$.

Each $B_n(i) - F$ is connected $(i = 1, 2, ..., n)$. And the number of edges between $B_n(i)$ and $B_n(j)$ is $(n-2)! > 2n-4(n \geq 6)$. Then $B_n - F$ is connected, a contradiction.

If $n = 4$, then $(n - 2)! = 2$ and $2n - 4 = 4$. There are at most two $[B_n(i), B_n(j)]$'s which are contained in F . Furthermore, $B_n - F$ has at most two components, a contradiction.

If $n = 5$ and there are i, j such that $|[B_n(i), B_n(j)]| = (n - 2)! = 2n - 4$, then$F = [B_n(i), B_n(j)]$. Hence $B_n - F$ is also connected, a contradiction.

Case 2. $n - 2 \leq |F_1| \leq 2n - 6$.

If each $B_n(i) - F$ is connected $(i = 1, 2, ..., n)$, then $B_n - F$ is connected. Hence there is some i such that $B_n(i) - F$ is not connected. And there are at most two subgraphs $B_n(i)$ and $B_n(j)$ which are not connected in $B_n - F$. From the inductive hypothesis, $B_n(i) - F$ and $B_n(j) - F$ have at most two components.

Subcase 2.1. Only one $B_n(i) - F_i$ is not connected.

Without loss of generality, we assume that $i = 1$. By the inductive hypothesis, $B_n(1) - F_1$ has two components.

Since $(n - 2)! \geq (2n - 4) - (n - 2)(n \geq 4)$, there are at most two subgraphs $B_n(i)$ and $B_n(j)$ such that $[B_n(i), B_n(j)] \subseteq F$. Thus the induced graph by $\bigcup_{i=2}^{n} B_n(i)$ is connected in $B_n - F$.

Furthermore, $|[B_n(1), B_n - B_n(1)]| = (n - 1)! > 2n - 4 - (n - 2)(n \geq 4)$. At least one component of $B_n(1) - F$ is connected to $\bigcup_{i=2}^{n} B_n(i)$. Hence, $B_n - F$ has at most two components, a contradiction.

Subcase 2.2. $B_n(1) - F_1$ and $B_n(2) - F_2$ are not connected in $B_n - F$.

It follows that $|E(B_n(1) \cap F)| = n - 2$ and $|E(B_n(2) \cap F)| = n - 2$. $B_n(i)(i = 1, 2)$ is super-λ by Proposition 2.3. Notice that $|F| \leq 2n - 4$. It is similar to Subcase 2.1, we can obtain $B_n - F$ is connected, a contradiction. We can see Figure 2.

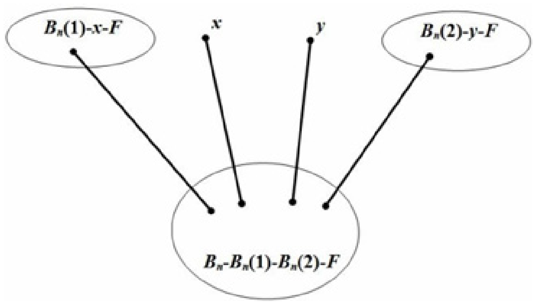

Figure 2. Illustration of subcase 2.2

Case 3. $2n - 5 \leq |F_1| \leq 2n - 4$.

Then $B_n(i) - F$ is connected for $i = 2, 3, ..., n$. Furthermore, $\bigcup_{i=2}^{n} B_n(i) - F$ is connected in $B_n - F$. Note that at most one vertex of $B_n(1)$ has no neighbors in $B_n - B_n(1)$. Hence, $B_n - F$ has at most two components, a contradiction.

Hence $\lambda^3(B_n) \geq 2n - 3$.

Theorem 2.6. $\lambda^4(B_n) = 3n - 5 (n \geq 3)$.

Proof. We prove the theorem by induction. It is true if $n = 3$. We assume that $n \geq 4$ and the result holds when $n \leq k - 1$. We will prove it for $n = k$.

Take a path $P = xyz$ and $F = E(x) \cup E(y) \cup E(z)$. Then $|F| = 3n-5$ and $B_n - F$ has at least four components. Hence $\lambda^4(B_n) \leq 3n - 5$. It suffices to show $\lambda^4(B_n) \geq 3n - 5$.

Suppose that there is an edge set F with $|F| \leq 3n - 6$, and $B_n - F$ has at least four components. Since B_n can be decomposed to n subgraphs $B_n(i)(i = 1, 2, ..., n)$, we set $F_i = E(B_n(i)) \cap F$ $(i = 1, 2, ..., n)$. Without loss of generality, we assume $|F_1| \geq |F_2| \geq ... \geq |F_n|$.

Case 1. $|F_1| \leq n - 3$.

Note that $\lambda(B_{n-1}) = n - 2$. Each $B_n(i) - F$ is connected $(i = 1, 2, ..., n)$. And the number of edges between $B_n(i)$ and $B_n(j)$ is $(n - 2)! > 3n - 6(n \geq 7)$. Then $B_n - F$ is connected, a contradiction.

If $n = 4$, then there exist at least one $[B_n(i), B_n(j)] \nsubseteq F$. Hence $B_n - F$ has at most three components, a contradiction.

If $n = 5$, then $(n - 2)! = 6$ and at most one $[B_n(i), B_n(j)]$ is contained in F . (If there is not $[B_n(i), B_n(j)] \subseteq F$, then $B_n - F$ is connected, a contradiction.) But $|[B_n(p), B_n(q)]| = (n - 2)! > 9 - 6$ for $p \neq i, j$ or $q \neq i, j$. Since each $B_n(i) - F$ is connected $(i = 1, 2, ..., n)$, $B_n - F$ is connected, a contradiction.

If $n = 6$, then $(n - 2)! = 12 = 3n - 6$ and there are i, j such that $[B_n(i), B_n(j)] = F$. Hence $B_n - F$ is also connected, a contradiction.

Case 2. $n - 2 \leq |F_1| \leq 3n - 8$.

If each $B_n(i) - F$ is connected $(i = 1, 2, ..., n)$, and $|[B_n(p), B_n(q)]| = (n - 2)! > 3n - 6 - (n - 2)(n \geq 6)$, then $B_n - F$ is connected, a contradiction. We assume $n = 5$. And there are some i, j such that $|[B_n(i), B_n(j)]| = (n - 2)! > 3n - 6 - (n - 2) = 6$. Hence $F = F_1 \cup [B_n(i),$

$B_n(j)$] and $B_n - F$ is also connected, a contradiction.

We suppose there are some i's such that $B_n(i) - F$ is not connected. But $\lambda(B_{n-1}) = n-2$, at most three subgraphs $B_n(i)$'s are not connected in $B_n - F$. From the inductive hypothesis, $B_n(i) - F$ has at most three components.

Subcase 2.1. Only one $B_n(i) - F_i$ is not connected in $B_n - F$.

First, assume $i = 1$. It is similar to Case 1, $\bigcup_{i=2}^{n} B_n(i)$ is connected in $B_n - F$. Since$|[B_n(1), B_n - B_n(1)]| = (n-1)! > 3n - 6 - (n-2)(n \geq 4)$. Hence, $B_n - F$ has at most three components, a contradiction.

Suppose that $i = 2$. Then $|F_1| \geq |F_2| \geq n - 2(i = 1, 2)$ and $\bigcup_{j=3}^{n} B_n(j) \cup B_n(1)$ is connected in $B_n - F$. Because $B_n(j) \cup B_n(1)$ is $|[B_n(2), B_n - B_n(2)]| = (n-1)! > 3n-6-(2n-4)(n \geq 4)$. Hence, $B_n - F$ has at most three components, a contradiction.

If $i \leq 3$, then $|F_1| \geq |F_2| \geq |F_i| \geq n - 2$. It follows that $|F_1| = |F_2| = |F_i| = n - 2$ and $F = F_1 \cup F_2 \cup F_i$. It is similar the above discussion, $B_n - F$ has at most three components, a contradiction.

Subcase 2.2. There are only two $B_n(i) - F_i$ and $B_n(j) - F_j$ that are not connected in $B_n - F$. And $i, j \leq 3$.

Subcase 2.2.1. Assume $i = 1, j = 2$.

Furthermore, we can obtain that $\bigcup_{j=3}^{n} B_n(i) - F$ is connected in $B_n - F$. And $|F_1| \leq 3n - 6 - (n-2) = 2n - 4$.

If $B_n(1) - F$ has three components, then $|F_1| \geq 2n - 5$ according to Theorem 2.5. But $|F_2| \geq n - 2$ and $|F| \leq 3n - 6$, at most one components of $B_n(i) - F$ $(i = 1, 2)$ is not connected to $\bigcup_{j=3}^{n} B_n(i) - F$. Then $B_n - F$ has at most two components, a contradiction. We can see Figure 3 and the dotted line represents that the component may be connected to another component.

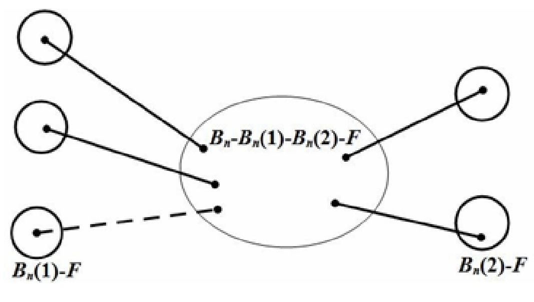

Figure 3. Illustration of $B_n(1) - F$ has three components

Hence we assume that $B_n(i) - F$ $(i = 1, 2)$ has only two components. Note that $|[B_n(i), B_n(j)]| = (n-2)! > 3n - 6 - (2n-4)(n \geq 4)$. At least one components of $B_n(i) - F$ $(i = 1, 2)$ is connected to $\bigcup_{i=3}^{n} B_n(i) - F$. Then $B_n - F$ has at most three components, a contradiction. We can see Figure 4 and the dotted lines represent that the component may be connected to another component.

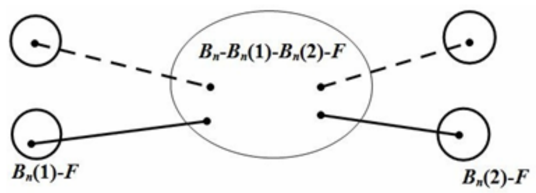

Figure 4. Illustration of $B_n(\text{i}) - F$ $(i = 1, 2)$ has only two components

Subcase 2.2.2. Assume $i = 1, j = 3$.

And $|F_1| \geq |F_2| \geq |F_3| \geq n - 2$. It follows that $F = F_1 \cup F_2 \cup F_3$ and $|F_1| = |F_2| = |F_3| = n - 2$. By Proposition 2.3, $B_n(i) - F$ $(i = 1, 3)$ is super-λ. Since $\bigcup_{i=3}^{n} B_n(i) - F$ is connected, $B_n - F$ is connected, a contradiction.

Subcase 2.2.3. $i = 2, j = 3$. It is similar to subcase 2.2.2.

Case 3. $3n - 7 \leq |F_1| \leq 3n - 6$.

Then $B_n(i) - F$ is connected for $i = 2, 3, ..., n$. Furthermore, $\bigcup_{i=2}^{n} B_n(i) - F$ is connected in $B_n - F$.

Note that at most one vertex of $B_n(1)$ has no neighbors in

$B_n - B_n(1)$. Hence, $B_n - F$ has at most two components, a contradiction. We can see Figure 5 and the dotted line represents that x may be connected to another component.

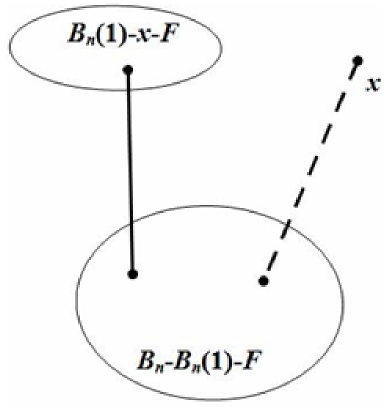

Figure 5. Illustration of case 3

Hence $\lambda^4(B_n) \geq 3n - 5$.

We can easily obtain $\kappa^3(B_n)$ for $n = 3$.

Theorem 2.7. $\kappa^3(B_3) = 3$.

Theorem 2.8. $\kappa^3(B_n) = 2n - 4(n \geq 4)$.

Proof. By induction. It is true if $n = 4$. We assume that $n \geq 5$ and the result holds when $n \leq k - 1$. We will prove it for $n = k$.

Take a cycle $C_4 = x_1x_2x_3x_4$ and $F = N(\{x_1, x_3\})$. Then $|F| = 2n-4$ and $B_n - F$ has at least three components. Hence $\kappa^3(B_n) \leq 2n - 4$. It suffices to show $\kappa^3(B_n) \geq 2n - 4$. Suppose that there is a vertex set F with $|F| \leq 2n - 5$, and $B_n - F$ has at least three components. Since B_n can be decomposed to n subgraphs $B_n(i)(i = 1, 2, ..., n)$, we set $S_1 = \{B_n(i) : |V(B_n(i)) \cap F| \geq n - 2\}$ and $S_2 = \{B_n(i) : |V(B_n(i)) \cap F| \leq n - 3\}$. Because $2(n-2) > 2n - 5 \geq |F|$, we can obtain $|S_1| \leq 1$ and $S_2 \neq \phi$. We obtain $S_2 - F$ is connected since $\kappa(B_{n-1}) = n - 2$. If $S_1 = \phi$, then $B_n - F$ is connected, a contradiction.

Hence $S_1 \neq \phi$. Assume $S_1 = \{B_n(1)\}$. If $B_n(1) - F$ is connected, then $B_n - F$ has at most two components, a contradiction.

Suppose $B_n(1)-F$ is not connected. Because $\kappa(B_n(1)) = n-2$ and $|V(B_n(1)) \cap F| \geq n-2$. If $B_n(1) - F$ has at least three components, then $|V(B_n(1)) \cap F| \geq 2n-6$ from the inductive hypothesis. Since $|F| \leq 2n-5$, at most one vertex of $B_n(1)$ has no neighbors in $S_2 - F$. Hence $B_n - F$ has at most two components, a contradiction. We can see Figure 5.

Suppose that $B_n(1) - F$ has only two components. The neighbors of $B_n(1)$ in S_2 is $(n-1)! > 2n-5 (n \geq 5)$. Then at least one component of $B_n(1) - F$ is connected to $S_2 - F$. Furthermore, $B_n - F$ has at most two components, a contradiction. We can see Figure 6 and the dotted line represents that the component may be connected to another component.

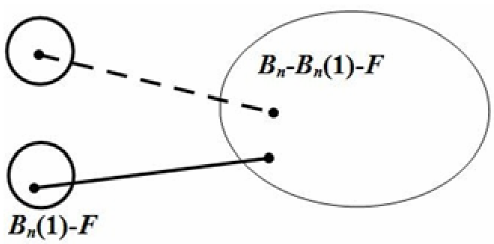

Figure 6. Illustration of $B_n(1) - F$ has only two components

Theorem 2.9. $\kappa^4(B_n) = 3n - 6 (n \geq 4)$.

Proof. By induction. It is true if $n = 4$. We assume that $n \geq 5$ and the result holds when $n \leq k - 1$. We will prove it for $n = k$.

Take a cycle $C_6 = x_1x_2 \ldots x_6$ and $A = \{x_1, x_3, x_5\}$. Then $|N(A)| = 3n - 6$ and $B_n - N(A)$ has at least four components. Hence $\kappa^4(B_n) \leq 3n - 6$. It suffices to show $\kappa^4(B_n) \geq 3n - 6$.

Suppose that there is a vertex set F with $|F| \leq 3n - 7$, and $B_n - F$ has at least four components. Since B_n can be decomposed to n subgraphs $B_n(i) (i = 1, 2, ..., n)$, we set $S_1 = \{B_n(i) : |V(B_n(i)) \cap F| \geq \lfloor (3n-7)/2 \rfloor + 1\}$ and $S_2 = \{B_n(i) : |V(B_n(i)) \cap F| \leq \lfloor (3n-7)/2 \rfloor\}$. Because $2(\lfloor (3n-7)/2 \rfloor + 1) > 3n - 7 \geq |F|$, we can obtain $|S_1| \leq 1$ and $S_2 \neq \phi$.

Case 1. $S_1 = \phi$.

Because of $\kappa(B_{n-1}) = n - 2$, there are at most two subgraphs $B_n(i)$ and $B_n(j)$ which are not connected in $S_2 - F$. Without loss of generality, assume $i = 1, j = 2$.

Subcase 1.1. There is only one $B_n(1)$ that is not connected in $S_2 - F$.

It follows that $|V(B_n(1)) \cap F| \geq n - 2$. From the inductive hypothesis, $\kappa^4(B_n(1)) = 3n - 9 > \lfloor (3n-7)/2 \rfloor$ $(n \geq 5)$. Hence $B_n(1) - F$ has at most three components. Note that each vertex of $B_n(t)$ has only one neighbor in $B_n - B_n(t)$. And the number of edges between $B_n(t)$ and $B_n(s)$ is $(n - 2)! > 2n - 5 (n \geq 5)$. Then $S^* = \bigcup_{i=2}^{n} B_n(i)$ is connected in $B_n - F$.

If $B_n(1) - F$ has three components, then $\kappa^3(B_n(1)) = 2n - 6$ by Theorem 2.8. And there are at most two isolated vertices in $B_n(1) - F$. Because $B_n(1)$ has $(n - 1)!$ neighbors in $B_n - B_n(1)$ and $(n - 1)! > 3n - 7 (n \geq 5)$, at least one component of $B_n(1) - F$ is connected to S^*. Furthermore, $B_n - F$ has at most three components, a contradiction. We can see Figure 7 and the dotted lines represent that the component may be connected to another component.

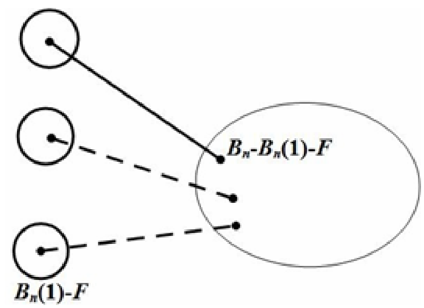

Figure 7. Illustration of $B_n(1) - F$ has three components

If $B_n(1) - F$ has only two components, then it is obviously a contradiction.

Subcase 1.2. There are only two subgraphs $B_n(1)$ and $B_n(2)$ that are not connected in $S_2 - F$.

It is similar to Subcase 1.1, $S^* = \bigcup_{i=3}^{n} B_n(i)$ is connected in $B_n - F$. We obtain $|V(B_n(1)) \cap F| \geq n - 2$ and $|V(B_n(2)) \cap F| \geq n - 2$. From the inductive hypothesis, $B_n(1) - F$ and $B_n(2) - F$ do not have more than three components, respectively. Because of the definition of S_2, $B_n(1) - F$ and $B_n(2) - F$ have only two components, respectively. Since the number of neighbors of $B_n(i)$ in S^* is $(n - 1)! - (n - 2)! > 3n - 7 (n \geq 5)$ for $i = 1, 2$. Then at least one component of $B_n(i) - F$ is connected to S^* for $i = 1, 2$. It follows that $B_n - F$ has at most three components, a contradiction. We can see Figure 4.

Case 2. $S_1 \neq \phi$. Assume $S_1 = \{B_n(1)\}$.

Because $|V(B_n(1)) \cap F| \geq \lfloor (3n-7)/2 \rfloor + 1$ and $|F| \leq 3n - 7$, at most one subgraph of S_2 is not connected.

Subcase 2.1. Each subgraph of S_2 is connected.

It is similar to Subcase 1.1, $S^* = \bigcup_{i=2}^{n} B_n(i)$ is connected in $B_n - F$.

If $B_n(1) - F$ has at least four components, then by induction, $\kappa^4(B_n(1)) = 3n - 9$. Because each vertex of $B_n(1)$ has only one neighbor in S^* and $|F| \leq 3n - 7$, at most two vertices of $B_n(1)$ are not connected to S^*. It follows that $B_n - F$ has at most three components, a contradiction.

If $B_n(1) - F$ has three components, then $\kappa^3(B_n(1)) = 2n - 6$. The number of neighbors of $B_n(i)$ in S^* is $(n - 1)! > 3n - 7 (n \geq 5)$. Then at least one component of

$B_n(1) - F$ is connected to S^*. It follows that $B_n - F$ has at most three components, a contradiction. We can see Figure 7.

If $B_n(1) - F$ has only two components, then obviously, it is a contradiction.

Subcase 2.2. Only one subgraph of S_2 is not connected. Suppose $B_n(2)$ is not connected.

Note that $S^* = \bigcup_{i=3}^{n} B_n(i)$ is connected. If $B_n(2) - F$ has at least three components, then $|V(B_n(2)) \cap F| + |V(B_n(1)) \cap F| > 3n - 7$ by Theorem 2.8 and $|V(B_n(1)) \cap F| \geq \lfloor (3n-7)/2 \rfloor + 1 (n \geq 5)$, a contradiction. Hence $B_n(2) - F$ has only two components and $|V(B_n(2)) \cap F| \geq n - 2$.

If $B_n(1) - F$ has at least four components, then by induction, $|V(B_n(1)) \cap F| \geq \kappa^4(B_n(1)) = 3n - 9$. But $n - 2 + 3n - 9 > 3n - 7 (n \geq 5)$, a contradiction.

If $B_n(1) - F$ has three components, then $|V(B_n(1)) \cap F| \geq \kappa^3(B_n(1)) = 2n - 6$. Note that $|V(B_n(2)) \cap F| \geq n - 2$, $|F| \leq 3n - 7$ and S^* is connected. After deleting at most one vertex, at least one component of $B_n(2) - F$ is connected to S^* and at least two components of $B_n(1) - F$ are connected to S^*. Hence $B_n - F$ has at most two components, a contradiction. We can see Figure 8 and the dotted line represents that the component may be connected to another component.

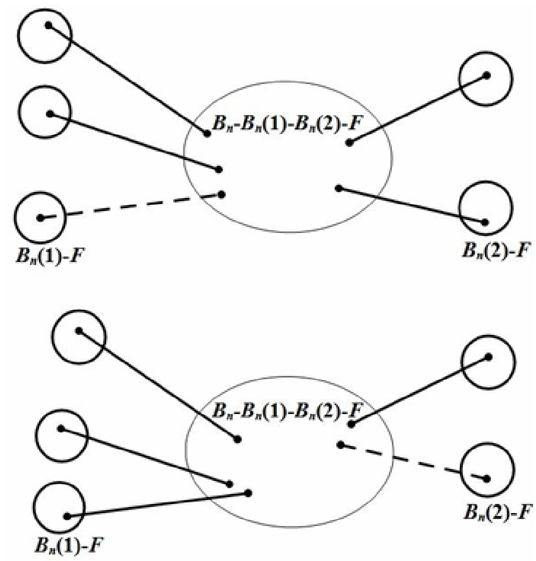

Figure 8. Illustration of $B_n(1) - F$ has three components

If $B_n(1) - F$ has only two components, then $|V(B_n(1)) \cap F| \geq n - 2$. Note that $|V(B_n(2)) \cap F| \geq n - 2$. Since the number of neighbors of $B_n(i)$ in S^* is $(n - 1)! - (n - 2)! > 3n - 7 (n \geq 5)$ for $i = 1, 2$. Then at least one component of $B_n(i) - F$ is connected to S^* for $i = 1, 2$. It follows that $B_n - F$ has at most three components, a contradiction. We can see Figure 9 and the dotted lines represent that the component may be connected to another component.

If $B_n(1) - F$ is connected, then obviously, it is a contradiction.

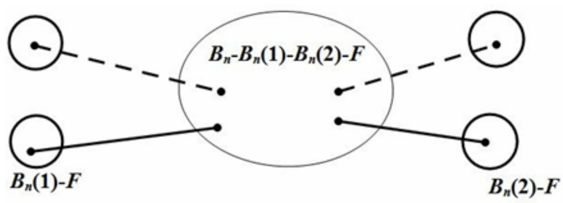

Figure 9. Illustration of $B_n(1) - F$ has two components

3 Conclusions

The component connectivity is a measurement of the reliability and fault-tolerant ability of networks. We determine the small component connectivity of bubble-sort networks. We will study the larger component connectivity of bubble-sort networks in the future work.

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