Strong Spanning Laceability of Mesh

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Abstract

A bipartite graph G is strong k^*-laceability if there is a r^*-container between any two distinct nodes x and y form different partite sets of G with $r \leq \min\{\deg(x), \deg(y), k\}$. The strong spanning laceability of G, $s\kappa_L^*(G)$, is the maximal value of G such that G is strong $s\kappa_L^*(G)$-connected and $s\kappa_L^*(G) \leq \Delta(G)$ where $\Delta(G)$ is the maximal degree of G. Let $M_{m,n}$ be the mesh with m rows and n columns. In this paper, we show that $s\kappa_L^*(M_{m,n}) = 3$ if mn is even and $\min\{m,n\} \geq 4$; otherwise $s\kappa_L^*(M_{m,n}) \leq 2$.

Keywords: One-to-one disjoint path cover, Strong spanning connectivity, Strong spanning laceability, Mesh

1 Introduction

The system performance of multi-processor networks, such as reliability and operating efficiency, has attracted widespread attention. The topological structure of a multi-processor network, such as diameter, connectivity, and disjoint paths, are all important indicators that affect system performance. Many scholars have done a lot of research in this area [3, 5-6, 14, 19].

Model this multi-processor interconnection network as a graph, where vertices and edges represent processors and communication channels, respectively. Routing is the process of transferring messages between vertices. The efficiency and reliability of routing plays a vital role in system performance. Obviously, when the amount of transmitted data is large, the internal vertex disjoint path can not only avoid congestion and speed up the transmission rate, but also provide optional transmission paths [7-8, 10]. When the vertex of the network fails, the vertex disjoint path can enhance the robustness of the system. All these advantages of disjoint paths are highlighted in the research of routing, reliability and fault tolerance in parallel and distributed systems [7-9, 11, 14].

A k-container of a graph G is a set of k disjoint paths between nodes u and v [6]. It is a k^*-container if it contains all nodes of G [2]. The k^*-container problem can be applied in many ways such as full utilization of network nodes, software testing, database design, and code optimization [1]. A graph G is k^*-connected if there exists a k^*-container between any two distinct nodes of G. The spanning connectivity of G, $\kappa^*(G)$, is the maximal value satisfies that G is k^*-connected for every $1 \leq k \leq \kappa^*(G)$. A graph G is super spanning connected if $\kappa^*(G) = \kappa(G)$ where $\kappa(G)$ is the connectivity of G.

A hamaltonian path is a path that contains all nodes, and a hamiltonian cychle is a cycle that contains all nodes. A graph G is hamiltonian graph if it contains a hamiltonian cycle. A bipartite graph G with partite sets X and Y is hamiltonian laceable if for any node x in X and for any node y in Y, there is a hamiltonian path of G between x and y.

A bipartite graph G is k^*-laceable if there exists a k^*-container between any two nodes from different partite sets of G. Obviously, a 2^*-container $\{P_1, P_2\}$ of graph G between x and y can combine to a hamiltonian cycle, and vice versa. Thus, a non-bipartite graph is a 2^*-connected if it is a hamiltonian graph, and a bipartite graph is 2^*-laceable if it is a hamiltonian graph. The spanning laceability of a bipartite graph G, $\kappa_L^*(G)$, is the maximal value satisfies that G is k^*-laceable for every $1 \leq k \leq \kappa_L^*(G)$. A bipartite graph G is super spanning laceable if $\kappa_L^*(G) = \delta(G)$.

There have been many research on the problem of

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super spanning connectivity and super spanning for many well-known networks [12-13, 17-18].

Let H be the graph with $V(H)=\{i\,|\,1\le i\le 6\}$ and $E(H)=\{\{j,k\}\,|\,1\le j<k<6\}-\{\{2,3\},\{5,6\}\}$ (see Figure 1). It is easy to see that $\deg(1)=\deg(4)=5$ and $\deg(j)=4$ for each $j\in\{2,3,5,6\}$. According to the symmetry of H, Table 1 shows that $\kappa^*(H)=4$ since $\kappa(H)=4$. It also shows that there is no 5^*-container between node 1 and node 2 since $\min\{\deg(1), \deg(2)\}=4$. And there is 5^*-container between node 1 and node 4. That is $\{\langle 1,4\rangle,\langle 1,2,4\rangle,\langle 1,5,4\rangle,\langle 1,3,4\rangle,\langle 1,6,4\rangle\}$. So, the definition of the spanning connectivity cannot describe the characteristics of k^*-container of H well. A graph G is strong k^*-connected if there is a r^*-container between any two distinct nodes x and y of G for each $r\le\min\{\deg(x),\deg(y),k\}$. The strong spanning connectivity of graph G, $s\kappa^*(G)$, is the maximal value of G such that G is strong connected and $s\kappa^*(G)\le\Delta(G)$ where $\Delta(G)$ is the maximal degree of G. Thus, we have $s\kappa^*(H)=5$..

Table 1. k^*-container of H between node x and y

$\{x,y\}$	$k=1$	$k=2$	$k=3$	$k=4$
$\{1,2\}$	$\{\langle 1,4,6,3,5,2\rangle\}$	$\{\langle 1,4,6,2\rangle,\langle 1,3,5,2\rangle\}$	$\{\langle 1,4,6,2\rangle,\langle 1,2\rangle,\langle 1,3,5,2\rangle\}$	$\{\langle 1,3,5,2\rangle,\langle 1,2\rangle,\langle 1,6,2\rangle,\langle 1,4,2\rangle\}$
$\{1,3\}$	$\{\langle 1,2,6,4,5,3\rangle\}$	$\{\langle 1,2,6,3\rangle,\langle 1,4,5,3\rangle\}$	$\{\langle 1,2,6,3\rangle,\langle 1,3\rangle,\langle 1,4,5,3\rangle\}$	$\{\langle 1,5,2,3\rangle,\langle 1,3\rangle,\langle 1,6,3\rangle,\langle 1,4,3\rangle\}$
$\{1,4\}$	$\{\langle 1,2,5,4,6,4\rangle\}$	$\{\langle 1,2,6,4\rangle,\langle 1,3,5,4\rangle\}$	$\{\langle 1,2,6,4\rangle,\langle 1,4\rangle,\langle 1,3,5,4\rangle\}$	$\{\langle 1,2,6,4\rangle,\langle 1,4\rangle,\langle 1,3,4\rangle,\langle 1,5,4\rangle\}$
$\{2,3\}$	$\{\langle 2,5,4,6,1,3\rangle\}$	$\{\langle 2,1,6,3\rangle,\langle 2,5,4,3\rangle\}$	$\{\langle 2,1,6,3\rangle,\langle 2,5,3\rangle,\langle 2,4,3\rangle\}$	$\{\langle 2,1,3\rangle,\langle 2,6,3\rangle,\langle 2,5,3\rangle,\langle 2,4,3\rangle\}$
$\{2,5\}$	$\{\langle 2,4,3,6,1,5\rangle\}$	$\{\langle 2,6,1,5\rangle,\langle 2,4,3,5\rangle\}$	$\{\langle 2,6,1,5\rangle,\langle 2,4,3,5\rangle,\langle 2,5\rangle\}$	$\{\langle 2,6,3,5\rangle,\langle 2,5\rangle,\langle 2,4,5\rangle,\langle 2,1,5\rangle\}$
$\{2,6\}$	$\{\langle 2,4,1,5,3,6\rangle\}$	$\{\langle 2,5,1,6\rangle,\langle 2,4,3,6\rangle\}$	$\{\langle 2,5,1,6\rangle,\langle 2,6\rangle,\langle 2,4,3,6\rangle\}$	$\{\langle 2,5,1,6\rangle,\langle 2,6\rangle,\langle 2,3,6\rangle,\langle 2,4,6\rangle\}$

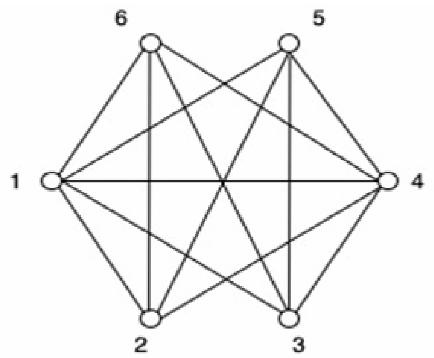

Figure 1. Graph H

Let Q be the bipartite graph with $V(Q)=\{i\,|\,1\le i\le 6\}$ and $E(Q)=\{\{i,j\}\,|\,i\in\{1,3,5\},j\in\{2,4,6\}\}-\{\{1,2\}\}$. (see Figure 2). It is easy to see that $\deg(1)=\deg(2)=2$ and $\deg(j)=3$ for each $j\in\{3,4,5,6\}$. Table 2 shows that $\kappa_L^*(Q)=2$ since $\Delta(Q)=3$. So, the definition of the spanning laceability cannot describe the characteristics of k^*-container of Q well. A graph G is strong k^*-laceable if there is a r^*-container between any two distinct nodes x and y from different partite sets of G for each $r\le\min\{\deg(x),\deg(y),k\}$. The strong spanning laceability of graph G, $s\kappa_L^*(G)$, is the maximal value of G such that G is strong $s\kappa_L^*(G)$-laceable and $s\kappa_L^*(G)\le\Delta(G)$ where $\Delta(G)$ is the maximal degree of G. Thus, we have $s\kappa_L^*(Q)=3$.

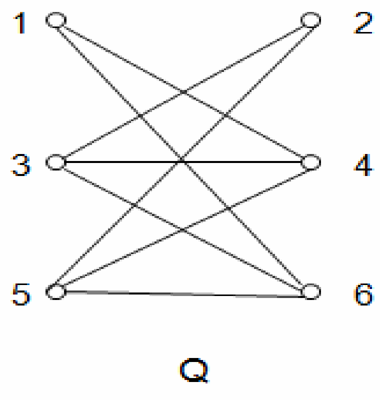

Figure 2. Graph Q

In this paper, we discuss the k^*-container problem of mesh. The rest of this paper is organized as follows. In Section 2, we introduce the definitions and notations for graph theory. In Section 3, we show that $s\kappa_L^*(M_{m,n})=3$ if mn is even and $\min\{m,n\}\ge 4$; otherwise $s\kappa_L^*(M_{m,n})\le 2$. Conclusion and future work are in the final section.

Table 2. k^*-container of Q between x and y

$\{x,y\}$	$k=1$	$k=2$	$k=3$
$\{1,2\}$	$\{\langle 1,4,3,6,5,2\rangle\}$	$\{\langle 1,4,3,2\rangle,\langle 1,6,5,2\rangle\}$	none
$\{1,4\}$	$\{\langle 1,6,3,2,5,4\rangle\}$	$\{\langle 1,4\rangle,\langle 1,6,3,2,5,4\rangle\}$	none
$\{3,4\}$	$\{\langle 3,2,5,6,1,4\rangle\}$	$\{\langle 3,2,5,4\rangle,\langle 3,6,1,4\rangle\}$	$\{\langle 3,2,5,4\rangle,\langle 3,4\rangle,\langle 3,6,1,4\rangle\}$

2 Definitions and Notations

For the definitions and notations of graph theory, we follow [4]. For any positive integer t, we set $[t]$ being the set $\{0,1,...,t-1\}$. A mesh $M_{m,n}$ with m rows and n columns, is a graph with $V(M_{m,n})=\{(i,j)\mid i\in[m], j\in[n]\}$ and $E(M_{m,n})=\{\{(i_1,j_1),(i_2,j_2)\}\mid i_1,i_2\in[m]$ and $j_1,j_2\in[n]$ such that either $i_1=i_2$ and $|j_1-j_2|=1$ or $j_1=j_2$, and $|i_1-i_2|=1\}$. Then $M_{m,n}$, is a bipartite graph with partite sets $X_{m,n}=\{(i,j)\in V(M_{m,n})\mid i+j \text{ is even}\}$ and $Y_{m,n}=\{(i,j)\in V(M_{m,n})\mid i+j \text{ is odd}\}$. We have $|X_{m,n}|=|Y_{m,n}|+1$ if both m and n are odd integers; otherwise, $|X_{m,n}|=|Y_{m,n}|$. We set $C_{m,n}=\{(0,0),(0,n-1),(m-1,0),(m-1,n-1)\}$, and set $X^*_{m,n}=X_{m,n}-C_{m,n}$ and $Y^*_{m,n}=Y_{m,n}-C_{m,n}$.

Let (a,c) and (b,d) be any two distinct nodes of $M_{m,n}$. We set $Q(a,b;c,d)$ being the path of $M_{m,n}$ joining (a,c) to (b,d) as follows:

$$Q(a,b:c,d)=\begin{cases}\langle(a,c),(a+1,c),...,(b,c)\rangle & \text{if } a<b \text{ and } c=d\\ \langle(a,c),(a-1,c),...,(b,c)\rangle & \text{if } a>b \text{ and } c=d\\ \langle(a,c),(a,c+1),...,(a,d)\rangle & \text{if } a=b \text{ and } c<d\\ \langle(a,c),(a,c-1),...,(a,d)\rangle & \text{if } a=b \text{ and } c>d\end{cases}.$$

3 Strong Spanning Laceability

3.1 1^*-container and 2^*-container

In this section, we will discuss 1^*-container and 2^*-container of mesh. Chen and Quimpo [15], and Itali [6] discusses the hamiltonian path properties of mesh as follows.

Theorem 3.1 [15] Let $x=(x_1,x_2)$ be any node in $X_{m,n}$ and let $y=(y_1,y_2)$ be any node in $Y_{m,n}$. If mn is even and $\{x,y\}\cap C_{m,n}\neq\varnothing$, then there is a 1^*-container in $M_{m,n}$ between x and y.

Theorem 3.2 [6] Let $x=(x_1,x_2)$ be any node in $X_{m,n}$ and let $y=(y_1,y_2)$ be any node in $Y_{m,n}$. (a) If $\min\{m,n\}\geq 4$ and mn is even, then there is a hamiltonian path in $M_{m,n}$ between x and y. (b) If $m\geq 3$, then there is a 1^*-container in $M_{m,n}$ between x and y except $1\leq x_1=y_1\leq m-2$, and $x_2=y_2-1$.

Obviously, $M_{2,2}$ is hamiltonian laceable. According to Theorem 3.1 and Theorem 3.2, we have the following result.

Corollary 3.3 Let m be any integer with $m\geq 2$ and let n be any even positive integer. For any node $x=(x_1,x_2)$ in $X_{m,n}$ and for any node $y=(y_1,y_2)$ in $Y_{m,n}$, there is a 1^*-container in $M_{m,n}$ between x and y except $m\geq 3, n=2, 1\leq x_1=y_1\leq m-2$, and $x_2=y_2-1$.

According to the definition of hamiltonian cycle and 2^*-container, a hamiltonian cycle of graph G and decompose to a 2^*-container between any two distinct nodes x and y in G Chen et al. [16] show that $M_{m,n}$ is hamiltonian if and only if mn is even and $\min\{m,n\}\geq 2$. Thus, we have the following theorem.

Theorem 3.4 Let m and n be any two integer with mn being even and $\min\{m,n\}\geq 2$. There is a 2^*-container in $M_{m,n}$ between any two distinct nodes x and y.

3.2 3^*-container

Here, we we discuss the 3^*-container of $M_{m,2n}$ for $m\geq 4$ and $n\geq 2$.

Lemma 3.5 For $m\geq 4$ and $n\geq 2$, let $B=\{(i,j)\mid i\in\{0,m-1\}$ and $j\in[2n]\}\cup\{(a,b)\mid a\in[m]$ and $b\in\{0,2n-1\}\}$. Let $x=(x_1,x_2)$ be any node in $X^*_{m,2n}\cap B$ and $y=(y_1,y_2)$ be any node in $Y^*_{m,2n}\cap B$. There is a 3^*-container in $M_{m,2n}$ between x and y.

Proof. For any node $z=(z_1,z_2)$ in B, we set $p(z)$ as $(1,z_2)$ if $z_1=0$, $(m-2,z_2)$ if $z_1=m-1$, $(z_1,1)$ if

$z_2 = 0$, and $(z_1, n-2)$ if $z_2 = n-1$. We set $C = \langle(0,0),(0,1),...,(0,2n-1),(1,2n-1),...,(m-1,2n-1)$ $(m-1,2n-2),...,(m-1,0),(m-2,0),...,(0,0)\rangle$.

Obviously, $\{x,y\} \subset V(C)$. Then we can decompose C into two disjoint paths P_1 *and* P_2 between x and y. Note that $M_{m,2n} - V(C)$ is isomprphic to $M_{m-2,2n-2}$. We have the following cases.

Case 1. $n = 2$.

Case 1.1 $x_1 \neq y_1$, $|x_2 - y_2| \neq 3$ or $\{x,y\} \subset \{\{(1,0),(1,3)\}, \{(m-2,0),(m-2,3)\}\}$. By Corollary 3.3, there is a hamiltonian path R in $M_{m,4} - V(C)$ joining $p(x)$ to $p(y)$. Then $\{P_1, P_2, P_3 = \langle x, p(x),\ R, p(y), y\rangle\}$ forms a 3^*-container in $M_{m,4}$ between x and y.

Case 1.2 $x_1 = y_1$, $|x_2 - y_2| = 3$, and $\{x,y\} \not\subset \{\{(1,0),(1,3)\},\{(m-2,0),(m-2,3)\}\}$. We set $u = (x_1, 0)$ and $v = (x_1, 3)$. Note that $\{u,v\} = \{x,y\}$. Then we set $R_1 = \langle u,(x_1,1),(x_1,2),v\rangle$, $R_2 = \langle u, Q(x_1-1,0:0,0), Q(0,x_1-1:1,1), Q(x_1-1,0:2,2), Q(0,x_1-1:3,3), v\rangle$, and $R_3 = \langle u, Q(x_1+1,m-1:0,0), Q(m-1,x_1+1,1:1), Q(x_1+1,m-1,2:2), Q(m-1,x_1+1,:3,3), v\rangle$. Then $\{R_1, R_2, R_3\}$ forms 3^*-container in $M_{m,4}$ between x and y.

Case 2. $n \geq 3$. We have $m-2 \geq 2$ and $2n-2 \geq 4$. By Corollary 3.3, there is a hamiltonian path R in $M_{m,2n} - V(C)$ joining $p(x)$ to $p(y)$. Then $\{P_1, P_2, P_3 = \langle x, p(x), R, p(y), y\rangle\}$, forms a 3^*-container in $M_{m,2n}$ between x and y.

Lemma 3.6 For $m \geq 4$ and $n \geq 2$, let $x = (x_1, x_2)$ be any node in $X^*_{m,2n}$ and $y = (y_1, y_2)$ be any node in $Y^*_{m,2n}$ with $x_1 = y_1$ or $x_2 = y_2$. There is a 3^*-container in between x and y.

Proof. We have the following cases.

Case 1. $x_1 = y_1$. By Lemma 3.5, there is a 3^*-container of $M_{m,2n}$ between x and y if $x_1 \in \{0, m-1\}$. Thus, we consider that $x_1 \notin \{0, m-1\}$. We set $\{p = (p_1, p_2), q = (q_1, q_2)\} = \{x, y\}$ such that $p_2 \leq q_2$. Note that $p_1 = q_1$. We have the following cases.

Case 1.1 $x_1 + 1 \neq m-1$. By Corollary 3.4, there is 2^*-container $\{P_1, P_2\}$ in $M_{x_1+1,2n}$ between p and q. By Corollary 3.3, there is a hamiltonian path R in $M_{m,2n} - M_{x_1+1,2n}$ between (x_1+1, x_2) and (y_1+1, y_2). Thus $\{P_1, P_2, P_3 = \langle p, R, q\rangle\}$ forms a 3^*-container between x and y.

Case 1.2 $x_1 + 1 = m-1$. By Corollary 3.3, there is a hamiltonian path R in $M_{x_1,2n}$ between (x_1-1, x_2) and (y_1-1, y_2). By Corollary 3.4, there is 2^*-container $\{P_1, P_2\}$ in $M_{m,2n} - M_{x_1,2n}$ between p and q. Thus $\{P_1, P_2, P_3 = \langle p, R, q\rangle\}$ forms a 3^*-container between x and y.

Case 2. $x_2 = y_2$. By Lemma 3.5, there is a 3^*-container of $M_{m,2n}$ between x and y if $x_2 \in \{0, 2n-1\}$. Thus, we consider that $x_2 \notin \{0, 2n-1\}$. Note that either $x_2 + 1$ is even or $2n - x_2$ is even. We set $\{p = (p_1, p_2), q = (q_1, q_2)\} = \{x, y\}$ such that $p_1 \leq q_1$. Note that $p_2 = q_2$.

Case 2.1 $x_2 + 1$ is even. Since $x_2 \notin \{0, 2n-1\}$ and $x_2 + 1$ is even, $x_2 \neq 2n-2$ and $2n - x_2 - 1 \geq 2$. By Corollary 3.4, there is 2^*-container $\{P_1, P_2\}$ in M_{m,x_2+1} between p and q. Note that $M_{m,2n} - (V(P_1) \cup V(P_2))$ is isomorphic to $M_{m,2n-x_2-1}$. By Corollary 3.3, there is a hamiltonian path R in $M_{m,2n} - (V(P_1) \cup V(P_2))$ between (p_1, p_2+1) and (q_1, q_2+1). Thus $\{P_1, P_2, P_3 = \langle p, R, q\rangle\}$ forms a 3^*-container between x and y.

Case 2.2 $2n - x_2$ is even. Since $x_2 \notin \{0, 2n-1\}$ and $2n - x_2$ is even, $x_2 \neq 1$ and $x_2 - 1 \geq 2$. By Theorem 3.2 and Theorem 3.3, there is hamiltonian path R in M_{m,x_2-1} between (p_1, p_2-1) and (q_1, q_2-1). Note that $M_{m,2n} - V(R)$ is isomorphic to $M_{m,2n-x_2}$. By Corollary 3.4, there is a 2^*-container $\{P_1, P_2\}$ in $M_{m,2n} - V(R)$ between p and q. Thus $\{P_1, P_2, P_3 = \langle p, R, q\rangle\}$ forms a 3^*-container between x and y.

Theorem 3.7 Let m be any positive integer with $m \geq 4$. Let $x = (x_1, x_2)$ be any node in $X^*_{m,4}$ and let $y = (y_1, y_2)$ be any node in $Y^*_{m,4}$. If there is a 3^*-container of $M_{m-2,4}$ between any node in $X^*_{m-2,4}$ and any node in $Y^*_{m-2,4}$, then there is a 3^*-container in $M_{m,4}$ between x and y.

Proof. By Lemma 3.6, there is a 3^*-container in $M_{m,4}$ between x and y if $x_1 = y_1$. Thus, we consider that $x_1 \neq y_1$. Without loss of generality, we assume that $x_1 < y_1$. We have the following cases.

Case 1. $y_1 \leq m-3$.

Case 1.1 $y \neq (m-3, 0)$. Since there is a 3^*-container in $M_{m-2,4}$ between any node $X^*_{m-2,4}$ and any node in $Y^*_{m-2,4}$, there is a 3^*-container $\{P_1, P_2, P_3\}$ of $M_{m-2,4}$ between x and

y. Since $\deg_{M_{m-2,4}}((m-3,0))=2$, we assume that $((m-3,0),(m-3,1))\in E(P_3)$. Thus, we write $P_3=\langle x,R_1,u,v,R_2,y\rangle$ where $\{u=(u_1,u_2),v=(v_1,v_2)\}=\{(m-3,0),(m-3,1)\}$. Let S be the subgraph induced by $\{(i,j)\mid i\in\{m-2,m-1\}$ *and* $j\in[4]\}$. Obviously, S is isomorphic to $M_{2,4}$. By Corollary 3.3, there is a hamiltonian path Z of S joining (u_1+1,u_2) to (v_1+1,v_2). Then $\{P_1,P_2,\langle x,R_1,u,Z,v,R_2,y\rangle\}$ forms a 3^*-container of $M_{m,4}$ between x and y.

Case 1.2 $y=(m-3,0)$. Since there is a 3^*-container of $M_{m-2,4}$ between any node in $X^*_{m-2,4}$ and any node in $Y^*_{m-2,4}$, there is a 3^*-container $\{P_1,P_2,P_3\}$ of $M_{m-2,4}$ between x and $(m-3,2)$. Since $\deg_{M_{m-2,4}}((m-3,0))=2$ and $\deg_{M_{m-2,4}}((m-3,3))=2$, we assume that $((m-3,0),(m-3,1))\in E(P_1)$ and $((m-3,2),(m-3,3))\in E(P_3)$. Without loss of generality, we assume that $P_3=\langle x,R,(m-3,3),(m-3,2)\rangle$. Obviously, combining P_1 and P_2 forms a cycle C which contains x and y. Then we can decompose C into two disjoint paths Z_1 and Z_2 between x and y such that $V(Z_1)\cup V(Z_2)=V(P_1)\cup V(P_2)$. By Corollary 3.3, there is a hamiltonian path Z of $M_{m,4}-M_{m-2,4}$ joining $(m-2,3)$ to $(m-2,0)$. Then $\{Z_1,Z_2,\langle x,R,(m-3,3),Z,y\rangle\}$ forms a 3^*-container in $M_{m,4}$ between x and y.

Case 2. $y_1\in\{m-2,m-1\}$ *and* $x_1\le m-4$. We have $y\in\{(m-2,0),(m-1,1),(m-2,2)\}$ if m is even, and $y\in\{(m-2,1),(m-1,2),(m-2,3)\}$ if m is odd.

Case 2.1 m is even. There is a 3^*-container $\{P_1,P_2,P_3\}$ of $M_{m-2,4}$ between x and $(m-3,1)$. Without loss of generality, we write $P_1=\langle x,R_1,(m-3,0),(m-3,1)\rangle$, $P_2=\langle x,R_2,(m-4,1),(m-3,1)\rangle$, and $P_3=\langle x,R_3,(m-3,2),(m-3,1)\rangle$. If $y=(m-2,0)$, then we set $Z_1=\langle x,R_1,(m-3,0),(m-2,0)\rangle$, $Z_2=\langle x,R_2,(m-4,1),(m-3,1),(m-2,1),(m-2,0)\rangle$, and $Z_3=\langle x,R_3,(m-3,2),(m-2,2),(m-2,3),(m-1,3),(m-1,2),(m-1,1),(m-1,0),(m-2,0)\rangle$. If $y=(m-1,1)$, then we set $Z_1=\langle x,R_1,(m-3,0),(m-2,0),(m-1,0),(m-1,1)\rangle$, $Z_2=\langle x,R_2,(m-4,1),(m-3,1),(m-2,1),(m-1,1)\rangle$, and $Z_3=\langle x,R_3,(m-3,2),(m-2,2),(m-2,3),(m-1,3),(m-1,2),(m-1,1)\rangle$. If $y=(m-2,2)$, then we set $Z_1=\langle x,R_1,(m-3,0),(m-2,0),(m-1,0),(m-1,1),(m-1,2),(m-1,3),(m-2,3),(m-2,2)\rangle$, $Z_2=\langle x,R_2,(m-4,1),(m-3,1),(m-2,1),(m-2,2)\rangle$, and $Z_3=\langle x,R_3,(m-3,2),(m-2,2)\rangle$. Then $\{Z_1,Z_2,Z_3\}$ forms a 3^*-container in $M_{m,4}$ between x and y.

Case 2.2 m is odd. There is a 3^*-container $\{P_1,P_2,P_3\}$ of $M_{m-2,4}$ between x and $(m-3,2)$. Without loss of generality, we write $P_1=\langle x,R_1,(m-3,1),(m-3,2)\rangle$, $P_2=\langle x,R_2,(m-4,2),(m-3,2)\rangle$, and $P_3=\langle x,R_3,(m-3,3),(m-3,2)\rangle$. If $y=(m-2,1)$, then we set $Z_1=\langle x,R_1,(m-3,1),(m-2,1)\rangle$, $Z_2=\langle x,R_2,(m-4,2),(m-3,2),(m-3,1),(m-2,1)\rangle$, and $Z_3=\langle x,R_3,(m-3,3),(m-2,3),(m-1,3),(m-1,2),(m-1,1),(m-1,0),(m-2,0),(m-2,1)\rangle$. If $y=(m-1,2)$, then we set $Z_1=\langle x,R_1,(m-3,1),(m-2,1),(m-2,0),(m-1,0),(m-1,1),(m-1,2)\rangle$, $Z_2=\langle x,R_2,(m-4,2),(m-3,2),(m-2,2),(m-1,2)\rangle$, and $Z_3=\langle x,R_3,(m-3,3),(m-2,3),(m-1,3),(m-1,2)\rangle$. If $y=(m-2,3)$, then we set $Z_1=\langle x,R_1,(m-3,1),(m-2,1),(m-2,0),(m-1,0),(m-1,1),(m-1,2),(m-1,3),(m-2,3)\rangle$, $Z_2=\langle x,R_2,(m-4,2),(m-3,2),(m-3,3)\rangle$, and $Z_3=\langle x,R_3,(m-3,3),(m-2,3)\rangle$. Then $\{Z_1,Z_2,Z_3\}$ forms a 3^*-container in $M_{m,4}$ between x and y.

Case 3. $y_1\in\{m-2,m-1\}$ and $x_1\ge m-3$. Let $\{P_1,P_2,P_3\}$ being a 3^*-container in $M_{m,4}-M_{2,4}$ between x and y. Since $\deg_{M_{m,4}-M_{2,4}}((2,0))=2$, we assume that $((2,0),(2,1))\in E(P_3)$. Thus, we write $P_3=\langle x,R_1,u,v,R_2,y\rangle$ where $\{u=(u_1,u_2),v=(v_1,v_2)\}=\{(2,0),(2,1)\}$. By Corollary 3.3, there is a hamiltonian path Z of $M_{2,4}$ joining (u_1-1,u_2) and (v_1-1,v_2). Then $\{P_1,P_2,\langle x,R_1,u,Z,v,R_2,y\rangle\}$ forms a 3^*-container in $M_{m,4}$ between x and y.

Lemma 3.8 For $m\in\{4,5\}$, let $x=(x_1,x_2)$ be any node in $X^*_{m,4}$ and $y=(y_1,y_2)$ be any node in $Y^*_{m,4}$. There is a 3^*-container in $M_{m,4}$ between x and y.

Proof. By Lemma 3.6, there is a 3^*-container in $M_{m,4}$ between x and y if $x_1=y_1$ *or* $x_2=y_2$. Thus, we consider that $x_1 \neq y_1$ *and* $x_2 \neq y_2$. Let $B_4 = \{(i,j) | i \in \{0,3\}$ and $j \in [4]\} \cup \{(a,b) | a \in [4]$ and $b \in \{0,3\}\}$ and $B_5 = \{(i,j) | i \in \{0,4\}$ and $j \in [4]\} \cup \{(a,b) | a \in [5]$ and $b \in \{0,3\}\}$. By Lemma 3.5, there is a 3^*-container of $M_{4,4}$ between x and y if $\{x,y\} \subset B_4$, and there is a 3^*-container of $M_{5,4}$ between x and y if $\{x,y\} \subset B_5$. Thus, we consider that $\{x,y\} \not\subset B_4$ if $m = 4$, and $\{x,y\} \not\subset B_5$ if $m = 5$.

Note that $x \in X^*_{4,4}$ and $y \in Y^*_{4,4}$ if $m = 4$, and $x \in X^*_{5,4}$ and $y \in Y^*_{5,4}$ if $m = 5$. According to the symmetric of $M_{4,4}$ and $M_{5,4}$, we consider that $x \in \{(0,2),(1,1),(2,0),(2,2),(1,3)\}$ if $m = 5$. Table 3 shows that there is a 3^*-container in $M_{m,4}$ between x and y for $m \in \{4,5\}$.

Hence there is a 3^*-container in $M_{m,4}$ between x and y for $m \in \{4,5\}$.

Table 3. 3^*-container of $M_{m,4}$ for $m \in \{4,5\}$

	x	y	3^*-container between x and y
$M_{4,4}$	$(0,2)$	$(2,1)$	$P_1 = \langle(0,2),(1,2),(2,2),(2,1)\rangle$, $P_2 = \langle(0,2),(0,1),(0,0),(1,0),(1,1),(2,1)\rangle$, $P_3 = \langle(0,2),(0,3),(1,3),(2,3),(3,3),(3,2),(3,1),(3,0),(2,0),(2,1)\rangle$
	$(1,1)$	$(2,3)$	$P_1 = \langle(1,1),(1,2),(2,2),(2,3)\rangle$, $P_2 = \langle(1,1),(0,1),(0,0),(0,1),(0,2),(0,3),(1,3),(2,3)\rangle$, $P_3 = \langle(1,1),(2,1),(2,0),(3,0),(3,1),(3,2),(3,3),(2,3)\rangle$
$M_{5,4}$	$(0,2)$	$(2,1)$	$P_1 = \langle(0,2),(1,2),(2,2),(3,2),(3,1),(2,1)\rangle$, $P_2 = \langle(0,2),(0,1),(0,0),(1,0),(1,1),(2,1)\rangle$, $P_3 = \langle(0,2),(0,3),(1,3),(2,3),(3,3),(4,3),(4,2),(4,1),(4,0),(3,0),(2,0),(2,1)\rangle$
	$(1,1)$	$(3,0)$	$P_1 = \langle(1,1),(2,1),(3,1),(3,0)\rangle$, $P_2 = \langle(1,1),(1,2),(0,2),(0,3),(1,3),(2,3),(2,2),(3,2),(3,3),(4,3),(4,2),(4,1),(4,0),(3,0)\rangle$, $P_3 = \langle(1,1),(0,1),(0,0),(1,0),(2,0),(3,0)\rangle$
	$(1,1)$	$(2,3)$	$P_1 = \langle(1,1),(1,2),(2,2),(2,3)\rangle$, $P_2 = \langle(1,1),(2,1),(2,0),(3,0),(4,0),(4,1),(3,1),(3,2),(4,2),(4,3),(3,3),(2,3)\rangle$, $P_3 = \langle(1,1),(1,0),(0,0),(0,1),(0,2),(0,3),(1,3),(2,3)\rangle$
	$(1,1)$	$(3,2)$	$P_1 = \langle(1,1),(2,1),(2,0),(3,0),(4,0),(4,1),(3,1),(3,2)\rangle$, $P_2 = \langle(1,1),(1,2),(2,2),(3,2)\rangle$, $P_3 = \langle(1,1),(1,0),(0,0),(0,1),(0,2),(0,3),(1,3),(2,3),(3,3),(4,3),(4,2),(3,2)\rangle$
	$(2,0)$	$(1,2)$	$P_1 = \langle(2,0),(2,1),(2,2),(1,2)\rangle$, $P_2 = \langle(2,0),(3,0),(4,0),(4,1),(3,1),(3,2),(4,2),(4,3),(3,3),(2,3),(1,3),(0,3),(0,2),(1,2)\rangle$, $P_3 = \langle(2,0),(1,0),(0,0),(0,1),(1,1),(1,2)\rangle$
	$(2,2)$	$(0,1)$	$P_1 = \langle(2,2),(1,2),(1,1),(0,1)\rangle$, $P_2 = \langle(2,2),(3,2),(3,3),(4,3),(4,2),(4,1),(4,0),(3,0),(3,1),(2,1),(2,0),(1,0),(0,0),(0,1)\rangle$, $P_3 = \langle(2,2),(2,3),(1,3),(0,3),,(0,2),(0,1)\rangle$

Table 3. 3^*-container of $M_{m,4}$ for $m \in \{4,5\}$ (continue)

	x	y	3^*-container between x and y
	$(2,2)$	$(1,0)$	$P_1 = \langle(2,2),(1,2),(1,1),(1,0)\rangle$, $P_2 = \langle(2,2),(2,3),(1,3),(0,3),(0,2),(0,1),(0,0),(1,0)\rangle$, $P_3 = \langle(2,2),(3,2),(1,3),(3,3),(4,3),(4,2),(4,1),(4,0),(3,0),(3,1),(2,1),(2,0),(1,0)\rangle$
$M_{5,4}$	$(1,3)$	$(2,1)$	$P_1 = \langle(1,3),(1,2),(1,1),(2,1)\rangle$, $P_2 = \langle(1,3),(2,3),(2,2),(3,2),(3,3),(4,3),(4,2),(4,1),(4,0),(3,0),(3,1),(2,1)\rangle$, $P_3 = \langle(1,3),(0,3),(0,2),(0,1),(0,0),(1,0),(2,0)(2,1)\rangle$
	$(1,3)$	$(3,2)$	$P_1 = \langle(1,3),(1,2),(2,2),(3,2)\rangle$, $P_2 = \langle(1,3),(0,3),(0,2),(0,1),(0,0),(1,0),(1,1),(2,1),(2,0),(3,0),(4,0),(4,1),(3,1),(3,2)\rangle$, $P_3 = \langle(1,3),(2,3),(3,3),(4,3),(4,2),(3,2)\rangle$,

Theorem 3.9 Let m be any integer with $m \geq 4$. For any node $x = (x_1, x_2)$ in $X^*_{m,4}$ and any node $y = (y_1, y_2)$ in $Y^*_{m,4}$, there is a 3^*-container in $M_{m,4}$ between x and y.

Proof. We proof this theorem by induction hypothesis on *m*. By Lemma 3.8, this theorem holds on $M_{m,4}$ for $m \in \{4,5\}$. Thus, we suppose that this theorem holds in $M_{k,4}$ for each $4 \leq k \leq m-1$ and $m \geq 4$. By Theorem 3.7, there is a 3^*-container in $M_{m,4}$ between x and y.

Theorem 3.10 Let *m* be any integer with $m \geq 4$, and let *n* be any integer with $n \geq 2$. For any node $x = (x_1, x_2)$ in $X^*_{m,2n}$ and for any node $y = (y_1, y_2)$ in $Y^*_{m,,2n}$, there is a 3^*-container in $M_{m,2n}$ between x and y.

Proof. We proof this theorem by induction hypothesis on *n*. By Lemma 3.9, this theorem holds on $M_{m,2n}$ for $n = 2$. Thus, we suppose that this theorem holds on $M_{m,2k}$ for each $2 \leq k \leq n-1$. By Lemma 3.6, there is a 3^*-container of $M_{m,2n}$ between x and y. if $x_2 = y_2$. Thus, we consider that $x_2 \neq y_2$. Without loss of generality, we assume that $x_2 < y_2$. We have the following cases.

Case 1. $y_2 \leq 2n-3$.

Case 1.1 $y \notin \{(0,2n-3),(m-1,2n-3)\}$. By induction hypothesis, there is a 3^*-container $\{P_1, P_2, P_3\}$ of $M_{m,2n-2}$ between x and y. Since $\deg_{M_{m,2n-2}}((0,2n-3)) = 2$, we assume that $((0,2n-3),(1,2n-3)) \in E(P_3)$. Thus, we write $P_3 = \langle x, R_1, u, v, R_2, y\rangle$ where $\{u = (u_1, u_2), v = (v_1, v_2)\} = \{(0,2n-3),(1,2n-3)\}$. By Corollary 3.3, there is a hamiltonian path Z of $M_{m,2n} - M_{m,2n-2}$ joining $(u_1, u_2 + 1)$ to $(v_1, v_2 + 1)$. Then $\{P_1, P_2, \langle x, R_1, u, Z, v, R_2, y\rangle\}$ forms a 3^*-container in $M_{m,2n}$ between x and y.

Case 1.2 $y = (0,2n-3)$. By induction hypothesis, there is a 3^*-container $\{P_1, P_2, P_3\}$ of $M_{m,2n-2}$ between x and $(2,2n-3)$. Without loss of generality, we assume that $P_1 = \langle x, R_1, (1,2n-3),(2,2n-3)\rangle$, $P_2 = \langle x, R_2, (2,2n-4),(2,2n-3)\rangle$, and $P_3 = \langle x, R_3, (3,2n-3),(2,2n-3)\rangle$. Since $\deg_{M_{m,2n-2}}(y) = 2$ and $(1,2n-3) \in V(P_1), y \in V(P_1)$. Thus, combining P_1 and P_2 forms a cycle C which contains x and y. Then we can decompose C into two disjoint paths Z_1 and Z_2 between x and y such that $V(Z_1) \cup V(Z_2) = V(P_1) \cup V(P_2)$. By Corollary 3.3, there is a hamiltonian path Z of $M_{m,2n} - M_{m,2n-2}$ joining $(3,2n-2)$ to $(0,2n-2)$. Then $\{Z_1, Z_2, \langle x, R_3, (3,2n-3), Z, y\rangle\}$ forms a 3^*-container in $M_{m,2n-2}$ between x and y.

Case 1.3 $y = \{m-1, 2n-3\}$. Similarly as Case 1.2, there is a 3^*-container in $M_{m,2n-2}$ between x and y.

Case 2. $y_2 \in \{2n-2, 2n-1\}$ and $x_2 \leq 2n-4$.

Case 2.1 $y_2 = 2n-2$ and $y_1 = 1$. By induction hypothesis, there is a 3^*-container $\{P_1, P_2, P_3\}$ of $M_{m,2n-2}$ between x and $(2,2n-3)$. Without loss of generally, we write $P_1 = \langle x, R_1, (1,2n-3),(2,2n-3)\rangle$, $P_2 = \langle x, R_2, (2,2n-4),(2,2n-3)\rangle$, and $P_3 = \langle x, R_3, (3,2n-3),(2,2n-3)\rangle$. Then we set $Z_1 = \langle x, R_1, (1,2n-3),(1,2n-2) = y\rangle$, $Z_2 = \langle x, R_2, (2,2n-4),(2,2n-3),(2,2n-2),(1,2n-2) = y\rangle$, and $Z_3 = \langle x, R_3, (3,2n-3),(3,2n-2),$

$..., (m-1, 2n-2), \ (m-1, 2n-1), (m-2, 2n-1)..., (0, 2n-1), (0, 2n-2), (1, 2n-2) = y\rangle$

Then $\{Z_1, Z_2, Z_3\}$ forms a 3^*-container in $M_{m,2n-2}$ between x and y.

Case 2.2 $y_2 = 2n-2$ and $y_1 > 1$. We have $y_1 \geq 3$ and y_1 is odd. By induction hypothesis, there is a 3^*-container $\{P_1, P_2, P_3\}$ of $M_{m,2n-2}$ between x and $(y_1-1, 2n-3)$. Without loss of generality, we write $P_1 = \langle x, R_1, (y_1-2, 2n-3), (y_1-1, 2n-3)\rangle$, $P_2 = \langle x, R_2, (y_1-1, 2n-4), (y_1-1, 2n-3)\rangle$ and $P_3 = \langle x, R_3, (y_1, 2n-3), (y_1-1, 2n-3)\rangle$

Then we set $Z_1 = \langle x, R_1, (y_1-2, 2n-3), (y_1-2, 2n-2), ..., (0, 2n-2), (0, 2n-1), ..., (m-1, 2n-1), (m-1, 2n-2), ..., (y_1, 2n-2) = y\rangle$, $Z_2 = \langle x, R_2, (y_1-1, 2n-4), (y_1-1, 2n--3), (y_1-1, 2n-2), (y_1, 2n-2) = y\rangle$, and $Z_3 = \langle x, R_3, (y_1, 2n-3), (y_1, 2n-2) = y\rangle$. Then $\{Z_1, Z_2, Z_3\}$ forms a 3^*-container in $M_{m,2n-2}$ between x and y.

Case 2.3 $y_2 = 2n-1$. By induction hypothesis, there is a 3^*-container $\{P_1, P_2, P_3\}$ of $M_{m,2n-2}$ between x and $(y_1, 2n-3)$. Without loss of generality, we write $P_1 = \langle x, R_1, (y_1-1, 2n-3), (y_1, 2n-3)\rangle$, $P_2 = \langle x, R_2, (y_1, 2n-4), (y_1, 2n-3)\rangle$ and $P_3 = \langle x, R_3, (y_1+1, 2n-3), (y_1, 2n-3)\rangle$. Then we set $Z_1 = \langle x, R_1, (y_1-1, 2n-3), (y_1-1, 2n-2), ..., (0, 2n-2), (1, 2n-2), ..., (y_1, 2n-2) = y\rangle$, $Z_2 = \langle x, R_2, (y_1, 2n-4), (y_1, 2n-3), (y_1, 2n-2) = y\rangle$, and $Z_3 = \langle x, R_3, (y_1+1, 2n-3), (y_1+1, 2n-2), ..., (m-1, 2n-2), \ (m-1, 2n-1), (m-2, 2n-1), ..., (y_1, 2n-2) = y\rangle$.

Then $\{Z_1, Z_2, Z_3\}$ forms a 3^*-container in $M_{m,2n-2}$ between x and y.

Case 3. $y_2 \in \{2n-2, 2n-1\}$ and $x_2 \geq 2n-3$. There is a 3^*-container $\{P_1, P_2, P_3\}$ of $M_{m,2n} - M_{m,2}$ between x and y. Since $\deg_{M_{m,2n}-M_{m,2}}((0,2)) = 2$, we assume that $((0,2),(1,2)) \in E(P_3)$. Thus, we write $P_3 = \langle x, R_1, u, v, R_2, y\rangle$ where $\{u = (u_1, u_2), v = (v_1, v_2)\} = \{(0,2), (1,2)\}$. By Corollary 3.3, there is a hamiltonian path Z of $M_{m,2}$ joining (u_1, u_2-1) to (v_1, v_2-1). Then $\{P_1, P_2, \langle x, R_1, u, Z, v, R_2, y\rangle\}$ forms a 3^*-container in $M_{m,2n}$ between x and y.

3.2 $s\kappa_L^*(M_{m,n})$

In this subsection, we discuss the strong spanning laceability of mesh. First, we discuss the 1^*-container of $M_{m,n}$ if mn is odd.

Lemma 3.11 For any node $x \in X_{m,n}$ and for any node $y \in Y_{m,n}$, there is no 1^*-container in $M_{m,n}$ between x and y if mn is odd.

Proof. It is easy to see that $|V(P) \cap X_{m,n}| = |V(P) \cap Y_{m,n}|$ for any path P of $M_{m,n}$ between x and y. Since $|X_{m,n}| = |Y_{m,n}| + 1$, there is no 1^*-container in $M_{m,n}$ between x and y.

According to the definition of the mesh and the definition of 1^*-container, we can obtain the following lemma.

Lemma 3.12 For $n \geq 1$, $s\kappa_L^*(M_{1,n}) = 0$ if $n \neq 2$ and $s\kappa_L^*(M_{1,n}) = 1$ if $n = 2$.

Lemma 3.13 For $m \geq 1$, $s\kappa_L^*(M_{m,2}) = 1$ if $m = 1$, $s\kappa_L^*(M_{m,2}) = 2$ if $m = 2$, and $s\kappa_L^*(M_{m,2}) = 0$ if $m \geq 3$.

Proof. By Lemma 3.12, $s\kappa_L^*(M_{1,2}) = 1$. Since $M_{2,2}$ is isomorphic to a cycle of length 4, it is easy to see that $s\kappa_L^*(M_{2,2}) = 2$. Thus, we consider that $m \geq 3$. Since $\kappa(M_{m,2}) = 2$ and $\{(1,0),(1,1)\}$ is a cut of set of $M_{m,2}$, there is no hamiltonian path of $M_{m,2}$ between $(1,0)$ and $(1,1)$. Thus, $s\kappa_L^*(M_{m,2}) = 0$ if $m \geq 3$.

Theorem 3.14 If mn is odd, then $s\kappa_L^*(M_{m,n}) = 0$. If n is even, then

$$s\kappa_L^*(M_{m,n}) = \begin{cases} 1, & \text{if } m = 1 \text{ and } n = 2 \\ 2, & \text{if } m = 2 \text{ and } n = 2 \\ 0, & \text{if } m = 3, \text{ or } m \leq 2 \text{ and } n \geq 4 \\ 3, & \text{if } m \geq 4 \text{ and } n \geq 4 \end{cases}.$$

Proof. By Lemma 3.11, $s\kappa_L^*(M_{m,n}) = 0$ if mn is odd. Thus, we consideer that mn is even. Without loss of generality, we assume that n is even. By Lemma 3.12, $s\kappa_L^*(M_{1,2n}) = 0$ if $n \neq 2$ and $s\kappa_L^*(M_{1,n}) = 1$ if $n = 2$. Thus, we assume that $m \geq 2$.

Case 1. $m = 2$. Since $M_{2,2}$ is isomorphic to a cycle of length 4, it is easy to see that $s\kappa_L^*(M_{2,2}) = 2$. Thus, we assume that $n \geq 4$. Since $M_{2,n}$ is isomorphic to $M_{n,2}$, by Lemma 3.13, $s\kappa_L^*(M_{2,n}) = 0$ if $n \geq 4$.

Case 2. $m = 3$. Let P be a path of $M_{3,n}$ between $(1,0)$ and $(1,1)$. Obviously, $(0,0) \notin V(P)$ or $(2,0) \notin V(P)$.

That is there is no hamiltonian path of $M_{3,n}$ between $(1,0)$ and $(1,1)$. Thus, $s\kappa_L^*(M_{3,n})=0$

Case 3. $m \geq 4$. By Lemma 3.13, $s\kappa_L^*(M_{m,2})=0$ if $m \geq 4$. Thus, we assume that $n \geq 4$. Let P_1, P_2, P_3, P_4 be four path of $M_{m,n}$ between $x=(1,1)$ and $y=(1,2)$. Obviously, $(0,0) \notin \bigcup_{i=1}^{4} V(P_i)$. That is there is not a 4^* -container in $M_{m,n}$ between *x* and *y*. Thus, $s\kappa_L^*(M_{m,n})<4$. By Corollary 3.3, Theorem 3.4, and Theorem 3.10, $s\kappa_L^*(M_{m,n})=3$.

4 Conclusion

In this paper, we propose the definition of strong spanning connectivity and strong spanning laceability which can describe the spanning connectivity or spanning laceability of general graphs more completely. Mesh is a non-regular graph. We show $s\kappa_L^*(M_{m,n})=3$ if mn is even, and $\min\{m,n\} \geq 4$; otherwise $s\kappa_L^*(M_{m,n}) \leq 2$. Next, we can study this topic on other non-regular graphs or regular graphs with fault elements.

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