# Structure Fault-tolerance of the Augmented Cube

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## Abstract

The augmented cube, denoted by $AQ_n$, is an important variant of the hypercube. It retains many favorable properties of the hypercube and possesses several embeddable properties that the hypercube and its other variations do not possess. Connectivity is one of the most important indicators used to evaluate a network's fault tolerance performance. Structure and substructure connectivity are the two novel generalizations of the connectivity, which provide a new way to evaluate fault-tolerant ability of a network. In this paper, the structure connectivity and substructure connectivity of the augmented cube for $H \in \{K_{1,M}, P_L, C_N\}$ is investigated, where $1 \le M \le 6, 1 \le L \le 2n-1$ and $3 \le N \le 2n-1$.

**Keywords:** Structure connectivity, Substructure connectivity, Fault-tolerant ability, Augmented cube, Interconnection network

## 1 Introduction

Fault-tolerant ability is a very important aspect for evaluating the performance of an interconnection network. An interconnection network with good fault-tolerant ability can run well and achieve ideal results even if some parts of the network fail or are damaged. Therefore, we hope the fault-tolerant ability of an interconnection network can be assessed by some indicators. Connectivity is one of the most important indicators we use to evaluate a network's fault-tolerant ability. A graph *G* with *n* vertices, after removing any $k-1$ vertices $(1 \le k < n)$, the resulting subgraph is still connected. After removing some *k* vertices, the graph *G* becomes a disconnected graph or a trivial graph. Then *G* is a *k*-connected graph, and *k* is called the *connectivity* of graph *G*, denoted by *k*(*G*). Generally, the larger the connectivity of a graph, the more stable the network it represents. Although the connectivity can correctly reflect the fault-tolerant performance of the system, it has an obvious drawback. That is, it assumes that all vertices adjacent to the same vertex will become faulty at the same time, and the probability of this case happening in real environment is very low. Hence, it does not accurately reflect the robust performance of large-scale networks. The conditional connectivity proposed by Harary [1] overcomes this shortcoming by attaching some requirements to each component when the entire network becomes disconnected due to failure of some vertices. Then, Fàbrega et al. [15] proposed the concept of *g*-extra connectivity. Given a graph *G* and a non-negative integer *g*, if there is a set of vertices in the graph *G* such that the graph *G* is disconnected after the vertex set is deleted and the number of vertices of each component is greater than *g*, then we call it a vertex cut. The minimum cardinality of all vertex cuts is referred to as the *g-extra commectivity* of graph *G*, denoted by $k_g(G)$. *g*-extra connectivity is a generalization of the superconnectivity. The *superconnectivity* of a graph *G* actually corresponds to $k_1(G)$ [15, 26]. More information on connectivity can be found in [4-14, 16, 18, 22-25].

However, both the connectivity and the improved conditional connectivity discussed above are based on the assumption that a single vertex failure is an independent event. Under such connectivities, when any vertex in the network fails, there is no effect on the vertices that are directly connected to this vertex. However, in fact, when a vertex in the network becomes faulty, the probability of vertices around this vertex will becoming faulty is greatly increased, which may form a faulty structure centered on this faulty vertex. Therefore, Lin et al. [2] proposed the concept of structure connectivity $k_g(Q_n, H)$ and sub-structure connectivity $k^s(Q_n, H)$ of the hypercube $Q_n$ in [2] for $H \in \{K_1, K_{1,1}, K_{1,2}, K_{1,3}, C_4\}$. They actually generalized the faulty element from a single faulty vertex to a faulty structure (substructure). More results on structure and substructure connectivity can be found in [17, 19-21, 27].

The augmented cube, proposed by Choudum and Sunitha [3], as an important variant of the hypercube, not only retains some of the superior properties of the

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hypercube, but also has many properties that are not available in hypercubes and other variants [28-29]. For example, the connectivity of the augmented cube is $2n-1$, which is almost twice that of a hypercube, This means that the fault tolerance ability of the augmented cube is somewhat higher than that of the hypercube. In this paper, we focus on the structure and substructure connectivity of augmented cube. We establish *H*-structure and *H*-substructure connectivity of $AQ_n$ for $H \in \{K_{1,M}, P_L, C_N\}$. (shown in Figure 1), respectively, where $1 \le k < 6$, $1 \le L < 2n-1$, and $3 \le N < 2n-1$.

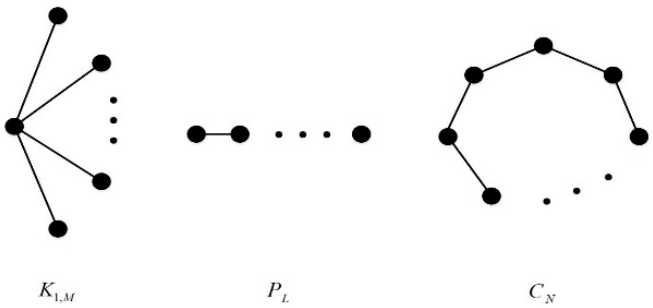



**Figure 1.** $K_{1,M}, P_L$ and $C_N$

The rest of the paper is structured as follows. In Section 2, the definition of the augmented cube and some useful properties of it are presented. Then, Section 3 presents the main results on $k(AQ_n, h)$ and $k^s(AQ_n, h)$ of augmented cube for each $H \in \{K_{1,M}, P_L, C_N\}$ in this paper. Conclusions are presented in Section 4.

## 2 Preliminaries

In order to better study the nature of the interconnection network, we generally model the interconnection network as an undirected graph, where each vertex in the graph represents a server, and each edge in the graph represents a communication link connecting two servers. A graph can be defined as a binary group: $G = (V(G), E(G))$, where: (1) $V(G)$ is a finite and nonempty set of vertices. (2) $E(G)$ is a finite set of edges connecting two different vertices (vertex pairs) in $V(G)$. In this paper, all graphs are referred to simple graphs. We use $N(u)$ to denote all vertices adjacent to the same vertex *u* for $u \in V(G)$.

For graphs *G* and *H*, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then *H* is called the subgraph of *G*, *G* is called the supergraph of *H*. If *H* is a subgraph of *G* with $V(H) = V(G)$, then *H* is called the spanning subgraph of *G*, and *G* is called the spanning supergraph of *H*. For the non-empty vertex subset $V' \in V(G)$ of graph *G*, if *G*'s subgraph *H* has $V'$ as its vertex set, and the two end vertices of each edge of *H* lie in $V'$, then subgraph *H* is called the induced subgraph of *G*. Two graphs *G* and *H* are isomorphic if there exists a bijection $f: V(G) \to V(H)$ such that $(u, v) \in E(G)$ if and only if $(f(u), f(v)) \in E(H)$. Let *H* be a subgraph in graph *G* and *F* be a set of elements and each element is a vertex subset of graph *G*. Let $W(F) = \bigcup_{s \in F} s$. If the set *F* satisfies that $G - W(F)$ is a disconnected graph or a trivial graph and the induced subgraph of each element in *F* is isomorphic to one of the spanning supergraphs of *H*, then *F* is called a *H-structure cut* of *G*. The *H-structure connectivity* of graph, denoted by $k(G, H)$, is the minimum cardinality of all *H*-structure cuts of *G*. If the induced subgraph of each element in *F* is isomorphic to one of the spanning supergraphs of a subgraph of *H*, then *F* is called a *H-substructure cut* of *G*. The *H-substructure connectivity* of graph *G*, denoted by $k^s(G, H)$, is the minimum cardinality of all *H*-substructure cuts of *G*. If *H* is just an isolated vertex. Then *H*-structure connectivity and *H*-substructure connectivity are exactly the traditional connectivity.

If the set of vertices of a graph *G* can be divided into two disjoint subsets *X* and *Y*, where $|S| = m$ and $|Y| = n$, such that any vertex in *X* has a unique edge with each vertex in *Y* and there is no edge has two end vertices in the same subset. Then *G* is called a complete bipartite graph, denoted by $K_{m,n}$. We use $K_1$ to represent an independent vertex. A *path* $P_k = \langle v_1, v_2, \ldots, v_k \rangle$ is a finite non-empty sequence with different vertices such that $(v_i, v_{i+1}) \in E(G)$ for $1 \le i \le k-1$. A *cycle* $C_k = \langle v_1, v_2, \ldots, v_k \rangle$ for $k \ge 3$ is a path where $(v_i, v_k) \in E(G)$.

In the following, we shall introduce the definition of the augmented cube and some properties of it.

**Definition 1.** *[3] Let an integer $n \ge 1$, an n-dimensional augmented cube $AQ_n$ consists of $2^n$ vertices, each vertex in $AQ_n$ is labeled by a unique n-bit binary string $u_n u_{n-1} \ldots u_2 u_1$, where $u_i \in \{0,1\}$ for $i = 1, 2, \ldots, n$. The augmented cube $AQ_1$ is a complete graph $K_2$ with two vertices 0 and 1. For $n \ge 2$, $AQ_n$ is build from two disjoint copies of $AQ_{n-1}$ according to the following steps: Let $AQ_{n-1}$ denote the graph obtained from one copy of $AQ_{n-1}$ by prefixing the label of each vertex with 0. Let $1AQ_{n-1}$ denote the graph obtained from the other copy of $AQ_{n-1}$ by prefixing the label of each vertex with 1. A vertex $u = 0u_{n-1} \ldots a_2 a_1$ of $0Q_{n-1}$ is adjacent to a vertex $v = 1b_{n-1} \ldots b_2 b_1$ of $1AQ_{n-1}$ if and only if, for $i = 1, \ldots, n-1$ either (1) $a_i = b_i$, in this case, $(u, v)$ is called a hypercube edge or (2) $a_i = \overline{b_i}$, in this case, $(u, v)$ is called a complement edge.*

For any vertex $u = a_{n-1}a_n \ldots a_1$ in augmented cube. we use $u^i$ (respectively, $\overline{u}^i$ ) to denote the binary string $a_n\, a_{i+1}\overline{a}_i a_{i-1} \ldots a_1$ (respectively, $a_n\, a_{i+1}\overline{a}_i\overline{a}_{i-1} \ldots \overline{a}_1$ ). It is clear that $u^1 = \overline{u}^1$, we may mix these two notations whenever it is convenient. For example, if $u = 011001$, then $u^1 = \overline{u}^1 = 011000$, $u^2 = 011011$, $u^4 = 010001$, $u^4 = 010110$, $(u^4)^2 = 010011$, $(u^4)^3 = 010010$, $(u^4)^2 = 010010$, and $(\overline{\overline{u}^4})^2 = 010101$.

The definition of augmented cube above is recursive. As with hypercube or other graphs, augmented cube also has several definitions. An alternative definition of $AQ_n$ is as follows:

**Definition 2.** *[3] An n-dimensional augmented cube with $n \ge 1$ contains $2^n$ vertices, each vertex of which is labeled by a unique n-bit binary string $u_n u_{n-1} \ldots u_2 u_1$, where $u_i\{0,1\}$ for $i = 1, 2, \ldots, n$. For any two vertices $a = a_n a_{n-1} \ldots a_2 a_1$ and $b = b_n b_{n-1} \ldots b_2 b_1$, a is adjacent to b, if and only if, there exists an integer k, $1 \le k \le n$, such that either (1) $a_k = \overline{b_k}$ and $a_i = b_i$ for $1 \le i \le n$, $i \ne k$ or (2) $a_i = \overline{b_i}$ for $1 \le i \le k$ and $a_i = b_i$ for $k+1 \le i \le n$.*

The augmented cubes $AQ_1, AQ_2$ and $AQ_3$ are shown in Figure 2.

Then, we give some properties of $AQ_n$.

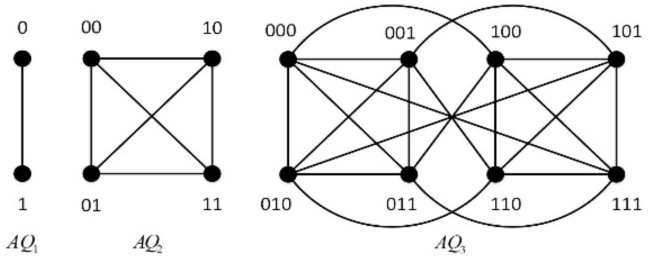



**Figure 2.** Augmented cubes $AQ_1, AQ_2$ and $AQ_3$ of dimension 1, 2, and 3

**Theorem 1.** *[3] $k(AQ_1) = 1$, $k(AQ_2) = 3$, $k(AQ_3) = 4$, and for $n \ge 4$, $k(AQ_1) = 2n-1$.*

According to Theorem 1, we have the following result.

**Theorem 2.** *For $n \ge 4$, $k(AQ_1, K_1) = k^s(AQ_n, K_1) = 2n-1$*

**Lemma 1.** *[26] For $n \ge 6$, $k_1(AQ_n) = 4n-8$.*

**Property 1.** *[26] If $(u, u^i)$ is a hypercube edge of dimension $i(1 \le i \le n)$, then*

$$N_{AQ_n}(u) \cap N_{AQ_n}(u^i) = \begin{cases} \{\overline{u}^i, \overline{u}^{i-1}\} & 2 \le i < n, \\ \{\overline{u}^2, u^2\} & i = 1. \end{cases}$$

That is, $u$ and $u^i$ have exactly two common neighbors in $AQ_n$ and $|N_{AQ_n}(\{u, u^i\})| = 4n-6$.

**Property 2.** *[26] If (u, $\overline{u}^i$ ) is a complement edge of dimension i $(2 \le i \le n)$, then*

$$N_{AQ_n}(u) \cap N_{AQ_n}(\overline{u}^i) = \begin{cases} \{\overline{u}^{i-1}, u^i, \overline{u}^{i+1}, \overline{u}^{i+1}\} & 2 \le i < n-1, \\ \{\overline{u}^{n-1}, u^n\} & i = n. \end{cases}$$

That is, $u$ and $\overline{u}^i$ have exactly four common neighbors in $AQ_n$ for $2 \le i \le n-1$ and $|N_{AQ_n}(\{u, \overline{u}^i\})| = 4n-8$. Similarly, $u$ and $\overline{u}^n$ have exactly two common neighbors in $AQ_n$ and $|N_{AQ_n}(\{u, \overline{u}^n\})| = 4n-6$.

**Property 3.** *[26] Any two vertices in $AQ_n$ have at most four common neighbors for $n \ge 3$.*

According to the Definition 1, we can easily obtain the following properties of augmented cube.

**Property 4.** *If $(u, u^i)$ and $(u, u^j)$ are two hypercube edges of dimensions i and $j(1 \le i \ne j \le n)$. Without loss of generality, we set $i < j$, then*

$$N_{AQ_n}(u^i) \cap N_{AQ_n}(\overline{u}^j) = \begin{cases} \{u, (u^i)^j, u^i, (u^j)^i\} & j = i+1 \text{ and } i > 1, \\ \{u, (u^i)^j\} & j > i+1, \\ \{u, (u^1)^2\} & i = 1 \text{ and } j = 2. \end{cases}$$

**Property 5.** *If $(u, \overline{u}^i)$ and $(u, \overline{u}^j)$ are two complement edges of dimensions i and $j(1 \le i \ne j \le n)$. Without loss of generality, we set I < j, then*

$$N_{AQ_n}(\overline{u}^i) \cap N_{AQ_n}(\overline{u}^j) = \begin{cases} \{u, \overline{u}^{i+1}, (u^j)^{j-1}, (u^j)^{j-1}\} & j = i+2, \\ \{u, (\overline{u}^j)^i\} & j = i+1 \text{ or } j > i+2. \end{cases}$$

**Property 6.** *If $(u, u^i)$ is a hypercube edge of dimension i and $(u, \overline{u}^j)$ is a complement edge of dimension $j(1 \le i, j \le n)$, then*

$$N_{AQ_n}(u^i) \cap N_{AQ_n}(\overline{u}^j) = \begin{cases} \{u^i, (u^i)^{i-1}, (\overline{u}^i)^{i-1}, (\overline{u}^i)^{i-1}\}, & i = j \text{ and } i > 1, \\ \{u, \overline{u}^i, (\overline{u}^j)^{i+1}, (u^i)^{i+1}\}, & i = j+1 \text{ and } i > n, \\ \{u, \overline{u}^i)^{i-1}\}, & i = n \text{ and } j > n-1, \\ \{u^i, (u^i)^{i-1}, (\overline{u}^i)^{i-1}, \overline{u}^{i-1}\}, & i = j+2, \\ \{u, \overline{u}^j\}, & |i-j| > 2, \\ \{u, (u^j)^i, (\overline{u}^j)^i, \overline{u}^i\}, & j = i+1 \text{ and } i > 1, \\ \{u, (u)^2\}, & i = 1 \text{ and } j = 2, \\ \{u, (\overline{u}^j)\}^i, & j = i+2 \text{ and } i > 1, \\ \{u, \overline{u}^2, (\overline{u}^3)^1, (u^1)^3, & i = 1 \text{ and } j = 3. \end{cases}$$

# 3 *H*-structure Connectivity and *H*-substructure Connectivity

In this section, we study the *H*-structure connectivity and *H*-substructure connectivity of $AQ_n$ for each $H \in \{K_{1,M}, P_L, C_N\}$, where $1 \le M < 6$, $1 \le L < 2n-1$, and $3 \le N < 2n-1$. Let *u* be an arbitrary vertex in $AQ_n$. In order to make the representation of our proof more convenient, we introduce a set of tokens $v^{[1]}, v^{[2]}, \ldots, v^{[2n-1]}$, where $v^{[1]}, v^{[2]}, \ldots, v^{[2n-1]}$ and all the adjacent vertices of *u*: $u^1, u^2, \overline{u}^2, \ldots, u^n, \overline{u}^n$ form a one-to-one correspondence. The correspondence of $u^j$ and $v^{[i]}$ is: (1) if *i* is even, then $j = \frac{i}{2} + 1$ and $u^j = v^{[i]}$. (2) if *i* is odd, then $j = \left\lfloor \frac{i}{2} \right\rfloor + 1$ and $\overline{u}^j = v^{[i]}$. The definition of $(v^{[i]})^l$ and $(\overline{v^{[i]}})^l$ is the same as that of $u^l$ and $\overline{u}^l$. For example, if $u = 000000$ is a vertex of $AQ_6$, then $u^4 = v^{[6]} = 001000$, $u^5 = v^{[9]} = 011111$ $(v^{[6]})^2 = 001010$ and $(\overline{v^{[9]}})^3 = 011000$. In this paper, we may mix these two notations whenever it is convenient.

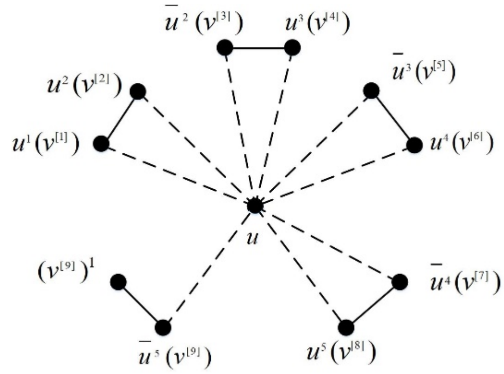
## 3.1 $k(AQ_n, K_{1,M})$ and $k^s(AQ_n, K_{1,M})$

According to the definition of $AQ_n$, Property 1, and Property 2, if *u* is an arbitrary vertex of $AQ_n$ and $(u, \overline{u}^i)$ is a complement edge of dimension $i (2 \le i \le n-1)$, then $N_{AQ_n}(u) \cap N_{AQ_n}(\overline{u}^i) = \{\overline{u}^{i-1}, u^i, u^{i+1}, \overline{u}^{i+1}\}$. The subgra-ph induced by $\{\overline{u}^i, \overline{u}^{i-1}, u^i, u^{i+1}, \overline{u}^{i+1}\}$ $(2 \le i \le n-1)$ is isomorphic to $K_{1,4}$. If $(u, \overline{u}^n)$ is a complement edge of dimension *n*, then $N_{AQ_n}(u) \cap N_{AQ_n}(\overline{u}^n) = \{\overline{u}^{n-1}, u^n\}$ and the subgraph induced by $\{\overline{u}^n, \overline{u}^{n-1}, u^n\}$ is isomorphic to $K_{1,2}$. Similarly, if $(u, u^i)$ is a hypercube edge of dimension $i (1 \le i \le n)$, then $N_{AQ_n}(u) \cap N_{AQ_n}(u^i) = \{\overline{u}^i, \overline{u}^{i-1}\}$ $(2 \le i \le n)$ and $N_{AQ_n}(u) \cap N_{AQ_n}(u^1) = \{u^2, \overline{u}^2\}$. The subgraph induced by $\{u^i, \overline{u}^i, \overline{u}^{i-1}\}$ $(2 \le i \le n)$ is isomorphic to $K_{1,2}$.

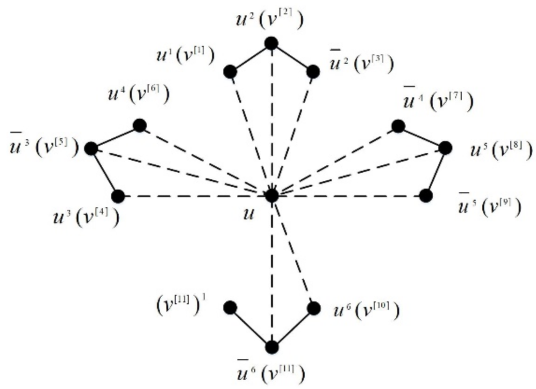
Here, we will discuss $k(AQ_n, K_{1,M})$ $k^s(AQ_n, K_{1,M})$ for the cases of $1 \le M \le 3$ and $4 \le M \le 6$.

### 3.1.1 $1 \le M \le 3$

**Lemma 2.** *For* $n \ge 4$ *and* $1 \le M \le 3$, $k(AQ_n, K_{1,M}) \le \left\lceil \frac{2n-1}{1+M} \right\rceil$ *and* $k^s(AQ_n, K_{1,M}) \le \left\lceil \frac{2n-1}{1+M} \right\rceil$.




(a) A $K_{1,1}$-structure-cut in $AQ_5$




(b) A $K_{1,2}$-structure-cut in $AQ_6$

**Figure 3.** A $K_{1,1}$-structure-cut in $AQ_5$ and a $K_{1,2}$-structure-cut in $AQ_6$

*Proof.* Let *u* be an any vertex in $AQ_n$. In the following, we distinguish cases for the values of *M* and *n*.

**Case 1.** $M = 1$. We set

$S_1 = \{\{v^{(M+1)i+1}, (v^{(M+1)i+2})\} \mid 0 \le i < \left\lfloor \frac{2n-1}{M+1} \right\rfloor\}$ and

$S_2 = \{\{v^{[2n-1]}, (v^{[2n-1]})\}^1\}$.

**Case 2.** $M = 2$.

**Case 2.1.** $n \equiv 0 \pmod 3$. We set

$$S_1 = \{\{v^{[(M+1)i+1]}, v^{[(M+1)i+2]}, v^{[(M+1)i+3]}\} \mid 0 \le i < \left\lfloor \frac{2n-1}{M+1} \right\rfloor\}$$

**Case 2.2.** $n \equiv 1 \pmod 3$. We set

$$S_1 = \{\{v^{[(M+1)]i+1]}, v^{[(M+1)i+2]}, v^{[(M+1)i+3]}\} \mid 0 \le i < \left\lfloor \frac{2n-1}{M+1} \right\rfloor\}$$

and $S_2 = \{\{v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2\}\}$.

**Case 2.3.** $n \equiv 2 \pmod 3$. We set

$$S_1 = \{\{v^{[(M+1)i+1]}, v^{[(M+1)i+2]}, v^{[(M+1)i+3]}\} \mid 0 \le i < \left\lfloor \frac{2n-1}{M+1} \right\rfloor\}$$

and $S_2 = \{\{v^{[2n-2]}, v^{[2n-2]}, (v^{[2n-1]})^2\}\}$.

**Case 3.** $M = 3$.
**Case 3.1.** $n \equiv 1 \pmod 4$. We set

$S_1 = \{\{v^{[(M+1)i+1]}, v^{[(M+1)i+2]}, v^{[(M+1)i+3]}, v^{[(M+1)i+4]}\} \mid 0 \le i < \left\lfloor \frac{2n-1}{M+1} \right\rfloor\}$ and $S_2 = \{\{v^{[(2n-1]}, (v^{[(2n-1]})^1, (v^{[(2n-1]})^2, (v^{[(2n-1]})^3\}\}$.

**Case 3.2.** $n \equiv 3 \pmod 4$. We set

$S_1 = \{\{v^{[(M+1)i+1]}, v^{[(M+1)i+2]}, v^{[(M+1)i+3]}, v^{[(M+1)i+4]}\} \mid 0 \le i < \left\lfloor \frac{2n-1}{M+1} \right\rfloor\}$ and $S_2 = \{\{v^{[(2n-3]}, v^{[(2n-2]}, v^{[(2n-1]}, (v^{[(2n-1]})^1\}\}$.

Suppose that $S = S_1$ when $M = 2$ and $n \equiv 0 \pmod 3$ ($S = S_1 \cup S_2$, otherwise). Clearly, if $M = 1$, the induced subgraph of each element in $S_1 \cup S_2$ is isomorphic to $K_{1,1}$; If $M = 2$, vertex $v^{[(M+1)i+2]}$ is adjacent to vertices $v^{[(M+1)i+1]}$ and $v^{[(M+1)i+3]}$ for $0 \le i < M\left\lfloor \frac{2n-1}{M+1} \right\rfloor$, vertex $v^{[2n-1]}$ is adjacent to vertices $v^{[2n-2]}$, $(v^{[2n-1]})^1$, and $(v^{[2n-1]})^2$. Therefore, the subgraph induced by each element in $S$ is isomorphic to $K_{1,2}$; If $M = 3$, vertex $v^{[(M+1)i+3]}$ is adjacent to vertices $v^{[(M+1)i+1]}$, $v^{[(M+1)i+2]}$, and $v^{[(M+1)i+4]}$ for $0 \le i < \left\lfloor \frac{2n-1}{M+1} \right\rfloor$, vertex $v^{[2n-1]}$ is adjacent to vertices $v^{[2n-3]}$, $v^{[2n-2]}$, $(v^{[2n-1]})$, $(v^{[2n-1]})^1$, $(v^{[2n-1]})^2$, and $(v^{[2n-1]})^3$. Thus, the subgraph induced by each element in $S$ is isomorphic to $K_{1,3}$. It is obvious that $|S| = \left\lceil \frac{2n-1}{1+M} \right\rceil$. Let $W(S) = \bigcup_{f \in S} f$. Since $AQ_n - W(S)$ is disconnected and one component of it is $\{u\}$, $k(AQ_n, K_{1,M}) \le \left\lceil \frac{2n-1}{1+M} \right\rceil$ and $k^s(AQ_n, K_{1,M}) \le \left\lceil \frac{2n-1}{1+M} \right\rceil$. Figure 3 shows a $K_{1,1}$-structure-cut in $AQ_5$ and a $K_{1,2}$-structure-cut in $AQ_6$.

**Lemma 3.** *For* $n \ge 4$ *and* $1 \le M \le 3$, $k^s(AQ_n, K_{1,M}) \ge \left\lceil \frac{2n-1}{1+M} \right\rceil$.

*Proof.* Let $F_n^*$ be a set of connected subgraphs in $AQ_n$, every element in the set is isomorphic to $K_{1,M}$ with $F_n^* \left\lceil \frac{2n-1}{1+M} \right\rceil - 1$. Hence $|V(F_n^*)| \le (1+M) \times (\left\lceil \frac{2n-1}{1+M} \right\rceil - 1) < 2n-1$. Since $k(AQ_n) = 2n-1$, $AQ_n - F_n^*$ is connected. The lemma holds.

Since $k(AQ_n, K_{1,M}) \ge k^s(AQ_n, K_{1,M})$, $k(Q_n, K_{1,M}) \ge \left\lceil \frac{2n-1}{1+M} \right\rceil$. By Lemma 2 and Lemma 3, we have the following theorem.

**Theorem 3.** *For* $n \ge 4$ *and* $1 \le M \le 3$, $k(AQ_n, K_{1,M}) = \left\lceil \frac{2n-1}{1+M} \right\rceil$ *and* $k^s(AQ_n, K_{1,M}) = \left\lceil \frac{2n-1}{1+M} \right\rceil$.

### 3.1.2 $4 \le M \le 6$

**Lemma 4.** *For* $n \ge 6$ *and* $4 \le M \le 6$, $k(AQ_n, K_{1,M}) \le \left\lceil \frac{n-1}{2} \right\rceil$ *and* $k^s(AQ_n, K_{1,M}) \le \left\lceil \frac{n-1}{2} \right\rceil$.

*Proof.* Let $u$ be an any vertex in $AQ_n$. In the following, we distinguish cases for the values of $M$ and $n$.
**Case 1.** $M = 4$.
**Case 1.1.** $n$ is odd. We set

$S_1 = \{\{v^{[1]}, v^{[2]}, v^{[3]}, v^{[4]}, v^{[5]}\}\}$ and

$S_2 = \{\{v^{[5+4i+1]}, v^{[5+4i+2]}, v^{[5+4i+3]}, v^{[5+4i+4]}, (v^{[5+4i+2]})^1\}, \mid 0 \le i < \left\lceil \frac{n-1}{2} \right\rceil - 1\}$

**Case 1.2.** $n$ is even. We set

$S_1 = \{\{v^{[1]}, v^{[2]}, v^{[3]}, v^{[4]}, v^{[5]}\}\}$,

$S_2 = \{\{v^{[5+4i+1]}, v^{[5+4i+2]}, v^{[5+4i+3]}, v^{[5+4i+4]}, (v^{[5+4i+2]})^1\}, \mid 0 \le i < \left\lceil \frac{n-1}{2} \right\rceil - 1\}$, and

$S_3 = \{\{v^{[2n-2]}, v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (v^{[2n-1]})^3\}\}$.

**Case 2.** $M = 5$.
**Case 2.1.** $n$ is odd. We set

$S_1 = \{\{v^{[1]}, v^{[2]}, v^{[3]}, v^{[4]}, v^{[5]}, (v^{[3]})^n\}\}$ and

$S_2 = \{\{v^{[5+4i+1]}, v^{[5+4i+2]}, v^{[5+4i+3]}, v^{[5+4i+4]}, (v^{[5+4i+2]})^1\}, (v^{[5+4i+2]})^2\} \mid 0 \le i < \left\lceil \frac{n-1}{2} \right\rceil - 1\}$.

**Case 2.2.** $n$ is even. We set

$S_1 = \{\{v^{[1]}, v^{[2]}, v^{[3]}, v^{[4]}, v^{[5]}, (v^{[3]})^n\}\}$,

$S_2 = \{\{v^{[5+4i+1]}, v^{[5+4i+2]}, v^{[5+4i+3]}, v^{[5+4i+4]}, (v^{[5+4i+2]})^1\}, (v^{[5+4i+2]})^2\} \mid 0 \le i < \left\lceil \frac{n-1}{2} \right\rceil - 1\}$, and

$S_3 = \{\{v^{[2n-2]}, v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (v^{[2n-1]})^3\}, (v^{[2n-1]})^4\}\}$.

**Case 3.** $M = 6$.
**Case 3.1.** $n$ is odd. We set

$S_1 = \{\{v^{[1]}, v^{[2]}, v^{[3]}, v^{[4]}, v^{[5]}, (v^{[3]})^n, (\overline{v^{[3]}})^n\}\}$ and

$S_2 = \{\{v^{[5+4i+1]}, v^{[5+4i+2]}, v^{[5+4i+3]}, v^{[5+4i+4]}, (v^{[5+4i+2]})^1\}, (v^{[5+4i+2]})^2, (v^{[5+4i+2]})^3\} \mid 0 \le i < \left\lceil \frac{n-1}{2} \right\rceil - 1\}$.

**Case 3.2.** $n$ is even. We set

$S_1 = \{\{v^{[1]}, v^{[2]}, v^{[3]}, v^{[4]}, v^{[5]}, (v^{[3]})^n, (\overline{v^{[3]}})^n\}\}$,

$S_2=\{\{v^{[5+4i+1]}, v^{[5+4i+2]}, v^{[5+4i+3]}, v^{[5+4i+4]}, (v^{[5+4i+2]})^1\}, (v^{[5+4i+2]})^2, (v^{[5+4i+2]})^3\} \mid 0 \le i < \left\lceil \frac{n-1}{2} \right\rceil - 1\}$, and

$S_3=\{\{v^{[2n-2]}, v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (v^{[2n-1]})^3\}, (v^{[2n-1]})^4\}, (v^{[2n-1]})^5\}\}$.

Obviously, the subgraph induced by the element in $S_1$ is isomorphic to $K_{1,M}$. For $0 \le i < \left\lfloor \frac{n-1}{2} \right\rfloor - 1$, vertex $v^{[5+4i+2]}$ is adjacent to vertices $v^{[5+4i+j]}$. with $j=1,3$ or $4$ and $(v^{[5+4i+2]})^p$ for $1 \le p \le 1+M-4$. Thus, the subgraph induced by each element in $S_2$ is isomorphic to $K_{1,M}$. Vertex $v^{[2n-1]}$ is adjacent to vertices $v^{[2n-2]}$ and $(v^{[2n-1]})^q$ for $1 \le p \le 1+M-(2n-2-4\times\left\lfloor \frac{n-1}{2} \right\rfloor)$. Therefore, the subgraph induced by the element in $S_3$ is isomorphic to $K_{1,M}$. Suppose that $S=S_1 \cup S_2$ when $n$ is odd ($S=S_1 \cup S_2 \cup S_3$, otherwise). Note that $|S|=\left\lfloor \frac{n-1}{2} \right\rfloor$. Let $W(S)=\bigcup_{f\in s} f$. Since $AQ_n-W(S)$ is disconnected and one component of it is $\{u\}$, $k(AQ_n, K_{1,M}) \le \left\lceil \frac{2n-1}{2} \right\rceil$ and $k^s(AQ_n, K_{1,M}) \le \left\lceil \frac{2n-1}{2} \right\rceil$ with $4 \le M \le 6$. Figure 4 shows a $K_{1,5}$-structure-cut of $AQ_6$.

**Lemma 5.** *Let $F_N$ be a $K_{1,M}$-substructure set of $AQ_n$ with $n \ge 6$ and $4 \le M \le 6$. If there exists an isolated vertex in $AQ_n - V(F_n)$, then $F_n \ge \left\lceil \frac{n-1}{2} \right\rceil$.*

*Proof.* Let $u$ be an any vertex in $AQ_n$. We set $W=\{x \mid (x,u)\}$ is a hypercube edge, $x \in V(G)\}$ and $Z=\{y \mid (y,u)$ is a complement edge, $y \in V(G)\}$. Clearly, $|W|=n$ and $|Z|=n-1$. By Property 2 and Property 3, each element in $F_N$ contains at most five distinct vertices in $N(u)$, namely, $\bar{u}^i, \bar{u}^{i-1}, \bar{u}^i, \bar{u}^{i+1}$, and $\bar{u}^{i+1}$ with $2 \le i \le n-1$. Since $\{(\bar{u}^i, \bar{u}^{i-1}), (\bar{u}^i, \bar{u}^i), (\bar{u}^i, \bar{u}^{i+1}), (\bar{u}^i, \bar{u}^{i+1})\} \subseteq E(AQ_n)$, the subgraph induced by $\{\bar{u}^i, \bar{u}^{i-1}, u^i, u^{i+1}, \bar{u}^{i+1}\}$ is isomorphic to $K_{1,4}$. We set $B=\{b_i \mid b_i \in F_n$ and $\{\bar{u}^{i-1}, \bar{u}^i, \bar{u}^i, u^{i+1}, \bar{u}^{i+1}\} \subseteq V(b_i) \cap N(u)\}$. Since each $V(b_i)$ contains three vertices in $Z$ and two vertices in $W$, $|B| < \left\lfloor \frac{2n-1}{5} \right\rfloor$. In the following, we distinguish cases for the value of $|B|$.

**Case 1.** $|B|=0$. Since each element in $F_N$ contains at most four distinct vertices in $N(u)$, $F_n \ge \left\lceil \frac{2n-1}{4} \right\rceil$.

**Case 2.** $|B|=1$. Suppose that $\{\bar{u}^{i-1}, u^i, \bar{u}^i, u^{i+1}\} \subseteq V(b_i)$ with $2 \le i \le n-1$. Since each element in $F_N - B$ contains at most four distinct vertices in $N(u)-V(B)$, $F_n \ge 1+\left\lceil \frac{2n-6}{4} \right\rceil = \left\lceil \frac{n-1}{2} \right\rceil$.

**Case 3.** $|B|=2$. Suppose that $\{\bar{u}^{i-1}, u^i, \bar{u}^i, u^{i+1}, \bar{u}^{i+1}\} \subseteq V(b_i), \{(\bar{u}^{j-1}, u^j, \bar{u}^j, u^{j+1}, \bar{u}^{j+1}\} \subseteq V(b_j)$ with $2 \le i, j \le n-1$, and $|i,j| \ge 3$. Without loss of generality, we set $j| \ge i$.

**Case 3.1.** $j-i=3$. Then $\{\bar{u}^{j-1}, u^j, \bar{u}^j, u^{j+1}, \bar{u}^{j+1}\} = \{\bar{u}^{i+2}, u^{i+3}, \bar{u}^{i+3}, u^{i+4}, \bar{u}^{i+4}\}$. Suppose that $w_1, w_2 \in N(u)-V(B)-\{u^{i+2}\}$ and $w_1 \ne w_2$. By Properties 4, 5 and 6, $N(u^{i+2}) \cap N(w_1) \cap N(w_2) = \phi$. In addition, vertex $u^{i+2}$ is not adjacent to the vertices in $N(u)-V(B)$. Therefore, there is an element $a \in F_N - B$ such that $u^{i+2} \in V(a)$ and $V(a) \cap \{(u)-V(B)\} \le 2$. Since each element in $F_N - B - \{a\}$ contains at most four distinct vertices in $N(u)-V(B)-V(a)$, $|F_n| \ge 2+1+\left\lceil \frac{2n-13}{4} \right\rceil = \left\lceil \frac{2n-1}{4} \right\rceil$.

**Case 3.2.** $j-i=4$. Then $\{\bar{u}^{j-1}, u^j, \bar{u}^j, u^{j+1}, \bar{u}^{j+1}\} = \{\bar{u}^{i+3}, u^{i+4}, \bar{u}^{i+5}, u^{i+5}, \bar{u}^{i+5}\}$. In the following, we distinguish cases for the number of elements containing three vertices $u^{i+2}, \bar{u}^{i+2}$ and $u^{i+3}$ in $F_n - B$. We use $S$ to denote the number of elements further deal with the following cases.

**Case 3.2.1.** $S=3$. Then there are three distinct elements in $N(u)-V(B)$ that contain one of three vertices $u^{i+2}, \bar{u}^{i+2}$, and $u^{i+3}$, respectively. Suppose that $w_1, w_2 \in N(u)-V(B)-\{u^{i+2}, \bar{u}^{i+2}, u^{i+3}\}$ and $w_1 \ne w_2$. By Properties 4, 5 and 6, $N(u^{i+2}) \cap N(w_1) \cap N(w_2) = \phi$. In addition, vertex $u^{i+2}$ is not adjacent to the vertices in $N(u)-V(B)-\{u^{i+2}, u^{i+3}\}$. Therefore, there is an element $a_1 \in F_n - B$ such that $u^{i+2} \in V(a_1)$ and $V(a_1) \cap \{N(u)-V(B)-\{\bar{u}^{i+2}, u^{i+3}\}\} \le 2$. For the cases of vertices $\bar{u}^{i+2}$ and $u^{i+3}$, the discussions are similar to that of vertex $u^{i+2}$ and we set $u^{i+2} \in a_2$, $u^{i+3} \in a_3$. Since each element in $F_N - B - \{a_1, a_2, a_3\}$ contains at most four distinct vertices in $N(u)-V(B)-V(a_1)-V(a_2)-V(a_3)$, $F_n \ge 2+3+\left\lceil \frac{2n-17}{4} \right\rceil = \left\lceil \frac{2n+3}{4} \right\rceil$.

**Case 3.2.2.** $S = 2$. Then there are two distinct elements in $N(u)-V(B)$, one of which contains one vertex in $u^{i+2}, \overline{u}^{i+2}$, and $u^{i+3}$ and another element contains the other two vertices. We assume that $a_1$ contains one vertex in $u^{i+2}, \overline{u}^{i+2}$ and $u^{i+3}$ and $a_2$ contains the other two vertices.

**Case 3.2.2.1.** $u^{i+2} \in V(a_1)$ and $\{\overline{u}^{i+2}, u^{i+3}\} \in V(a_2)$. Similar to the discussion of Case 3.2.1, we have $V(a_1) \cap \{N(u)-V(B)-\{u^{i+2}, u^{i+3}\}\} \leq 2$. Since $N(\overline{u}^{i+2}) \cap N(u^{i+3}) - \{u, \overline{u}^{i+3}\} = \{(\overline{u}^{i+2})^{i+4}, (\overline{u}^{i+3})^{i+4}\}$, each element in $\{(\overline{u}^{i+2})^{i+4}, (\overline{u}^{i+3})^{i+4}\}$ is not adjacent to the vertices in $N(u)-V(B)-\{u^{i+2}, \overline{u}^{i+2}, \overline{u}^{i+3}\}$ and each element in $\{u^{i+2}, u^{i+3}\}$ is not adjacent to the vertices in $N(u)-V(B)-\{u^{i+2}\}$, $V(a_2) \cap \{N(u)-V(B)-\{u^{i+2}\}\} = 2$ and $\{\overline{u}^{i+2}, u^{i+3}\} \in V(a_2)$. Since each element in $F_N - B\{a_1, a_2\}$ contains at most four distinct vertices in $N(u)-V(B)-V(a_1)-V(a_2)$, $|F_n| \geq 2+1+1+\left\lceil \frac{2n-15}{4} \right\rceil = \left\lceil \frac{2n+1}{4} \right\rceil$. For the case of vertices $u^{i+3} \in V(a_1)$ and $\{\overline{u}^{i+2}, u^{i+2}\} \in V(a_2)$, the discussion is similar.

**Case 3.2.2.2.** $\overline{u}^{i+2} \in V(a_1)$ and $\{u^{i+2}, u^{i+3}\} \in V(a_2)$. Similar to the discussion of Case 3.2.1, we have $V(a_1) \cap \{N(u)-V(B)-\{u^{i+2}, u^{i+3}\}\} \leq 2$. Since $N(\overline{u}^{i+2}) \cap N(u^{i+3}) - \{u, \overline{u}^{i+2}\} = \{(u^{i+2})^{i+3}, (u^{i+3})^{i+3}\}$, each element in $\{(u^{i+2})^{i+3}, (u^{i+3})^{i+2}\}$ is not adjacent to the vertices in $N(u)-V(B)-\{\overline{u}^{i+2}\}$ and $(u^{i+2}, u^{i+3}) \notin E(AQ_n)$, $V(a_2) \cap \{N(u)-V(B)-\{u^{i+2}\}\} \leq 2$ and $\{u^{i+2}, u^{i+3}\} \subseteq V(a_2)$. Since each element in $F_n - B - \{a_1, a_2\}$ contains at most four distinct vertices in $N(u)-V(B)-V(a_1)-V(a_2)$, $|F_n| \geq 2+1+1+\left\lceil \frac{2n-15}{4} \right\rceil = \left\lceil \frac{2n+1}{4} \right\rceil$.

**Case 3.2.3.** $S = 1$. According to the discussions of Case 3.2.1 and Case 3.2.2, there is an element $a_1 \in F_n - B$ such that $V(a) \cap \{N(u)-V(B) = 3$ and $\{u^{i+2}, \overline{u}^{i+2}, u^{i+3},\} \subseteq V(a)$. Since each element in $F_n - B - \{a\}$ contains at most four distinct vertices in $N(u)-V(B)-V(a)$, $|F_n| \geq 2+1+\left\lceil \frac{2n-14}{4} \right\rceil = \left\lceil \frac{n-1}{2} \right\rceil$.

**Case 3.3.** $j-i \geq 5$. We set $U_{ij} = \{u^{i+2}, \overline{u}^{i+2}, ..., u^{j-1}\}$. In the following, we will calculate the number of elements containing $U_{ij}$ in $F_n - B$. Since $5 \leq |U_{ij}| \leq 2n-11$ and each element contains at most four distinct vertices of $N(u)-V(B)$ in $F_n - B$, we will distinguish cases for the value of $|U_{ij}|$.

**Case 3.3.1.** $|U_{ij}| \equiv 1$ (mod 4). Similar to the discussion in Case 3.1, we have $|F_n| \geq 2+1+\left\lceil \frac{2n-13}{4} \right\rceil = \left\lceil \frac{2n-1}{4} \right\rceil$.

**Case 3.3.2.** $|U_{ij}| \equiv 3$ (mod 4). Similar to the discussion in Case 3.2 we have $|F_n| \geq 2+1+\left\lceil \frac{2n-14}{4} \right\rceil = \left\lceil \frac{n-1}{2} \right\rceil$.

**Case 4.** $|B| \geq 3$. If $b_i, b_j \in B$ and there is no $b_k \in B$ with $i < k < j$, we set $U_{ij} = \{u^{i+2}, \overline{u}^{i+2}, ..., u^{i-1}\}$. According to the discussion of Case 3, if $|U_{ij}| \equiv 3$ (mod 4), then the value of $F_n$ will be the smallest. Thus, $|F_n| \geq |B| + (|B|-1) + \left\lceil \frac{2n-1-5\times|B|-3\times(|B|-1)}{4} \right\rceil = \left\lceil \frac{n-1}{2} \right\rceil$.

In summary, the lemma holds.

**Lemma 6.** *For* $n \geq 6$ *and* $4 \leq M \leq 6$, $k(AQ_n, K_{1,M}) \geq \left\lceil \frac{n-1}{2} \right\rceil$ *and* $k^s(AQ_n, K_{1,M}) \geq \left\lceil \frac{n-1}{2} \right\rceil$.

*Proof.* We will prove this lemma by contradiction. Let $F_n^*$ be a $K_{1,M}$-substructure set of $AQ_n$ and $|F_n^*| \leq \left\lceil \frac{n-1}{2} \right\rceil - 1$. If $AQ_n - V(F_n^*)$ is disconnected, then we let $R$ be the smallest component of $AQ_n - V(F_n^*)$. Note that $|V(F_n^*)| \leq (1+M) \times (\left\lceil \frac{n-1}{2} \right\rceil - 1) \leq 7 \times (\left\lceil \frac{n-1}{2} \right\rceil - 1)$. By Lemma 1, we have $7 \times (\left\lceil \frac{n-1}{2} \right\rceil - 1 < 4n-8$ for $n \geq 6$. Hence $|V(R)| = 1$. Furthermore, we assume that vertex $u \in V(R)$. By Lemma 5, $|N(u) \cap V(F_n^*)| \leq 2n-1 < 2n-1$, which means that there exists at least one neighbor of $u$ in $AQ_n - V(F_n^*)$. Therefore, we have $|V(R)| \geq 2$, a contradiction. Thus, $AQ_n - V(F_n^*)$ is connected. The lemma holds.

Combining Lemma 4, we have $k^s(AQ_n, K_{1,M}) = \left\lceil \frac{n-1}{2} \right\rceil$. Since $k(Q_n, K_{1,M}) \geq k^s(Q_n, K_{1,M})$, $k(Q_n, K_{1,M}) \geq \left\lceil \frac{2n-1}{1+M} \right\rceil$. By Lemma 4 and Lemma 6, we have the following theorem.

**Theorem 4.** *For* $n \geq 6$ *and* $4 \leq M \leq 6$, $k(AQ_n, K_{1,M}) = \left\lceil \frac{n-1}{2} \right\rceil$ *and* $k^s(AQ_n, K_{1,M}) = \left\lceil \frac{n-1}{2} \right\rceil$.

### 3.2 $k(AQ_n, P_L)$ and $k^s(AQ_n, P_L)$

Let $u$ be an arbitrary vertex in $AQ_n$, according to the definition of $AQ_n$, $(\overline{u}^{i-1}, u^i) \in E(AQ)$ and $(u^i, \overline{u}^i) \in E(AQ)$ $(2 \le i \le n)$. Thus $<(u^1, u^2, \overline{u}^2, ..., u^n, \overline{u}^n)$ can form a path with length of $2n-2$.

Since $P_2(P_3)$ is isomorphic to $K_{1,1}(K_{1,2})$ and we have given $k(AQ_n, K_{1,1})(k(AQ_n, K_{1,2}))$ and $k^s(AQ_n, K_{1,1})$ $(k^s(AQ_n, K_{1,2}))$ in section 3.1, we assume $L \ge 4$ in the following.

**Lemma 7.** *For* $n \ge 3$ *and* $4 \le L \le 2n-1$, $k(AQ_n, P_L) \le \left\lceil \frac{2n-1}{L} \right\rceil$ *and* $k^s(AQ_n, P_L) \le \left\lceil \frac{2n-1}{L} \right\rceil$.

*Proof.* Let $u$ be an arbitrary vertex in $AQ_n$. We set $S_1 = \{\{v^{[i \times L+1]}, v^{[i\times L+2]}, ..., v^{[i\times L+L]}\} \mid 0 \le i < \left\lfloor \frac{2n-1}{L} \right\rfloor\}$.

If $(2n-1) \equiv 0 \pmod L$, then we set $S_2 = \phi$. Otherwise, according to the values of $L$ and $\left\lceil \frac{2n-1}{L} \right\rceil \times L - 2n+1$, we will divide into the following cases,

**Case 1.** $L$ is odd and $L \le 2n-3$.

**Case 1.1.** $\left\lceil \frac{2n-1}{L} \right\rceil \times L - 2n+1$ is even. We set $S_2 = \{\{v^{[\left\lfloor \frac{2n-1}{L} \times L+1 \right\rfloor]}, ..., v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (\overline{v^{[2n-1]}})^1, (v^{[2n-1]})^3, (\overline{v^{[2n-1]}})^3, ..., (v^{[2n-1]})^{\frac{\left\lceil \frac{2n-1}{L} \right\rceil \times L-2n+1]}{2}+1}\}\}$.

**Case 1.2.** $\left\lceil \frac{2n-1}{L} \right\rceil \times L - 2n+1$ is odd. We set $S_2 = \{\{v^{[\left\lfloor \frac{2n-1}{L} \times L+1 \right\rfloor]}, ..., v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (\overline{v^{[2n-1]}})^1, (v^{[2n-1]})^3, (v^{[2n-1]})^3, (\overline{v^{[2n-1]}})^3, ..., (\overline{v^{[2n-1]}})^{\frac{\left\lceil \frac{2n-1}{L} \right\rceil \times L-2n]}{2}+1}\}\}$.

**Case 2.** $L$ is even and $L \le 2n-4$. We set $S_2 = \{\{v^{[\left\lfloor \frac{2n-1}{L} \times L+1 \right\rfloor]}, ..., v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (\overline{v^{[2n-1]}})^2, (v^{[2n-1]})^3, (\overline{v^{[2n-1]}})^3, ..., (\overline{v^{[2n-1]}})^{\frac{\left\lceil \frac{2n-1}{L} \right\rceil \times L-2n}{2}+1}\}\}$.

**Case 3.** $L = 2n-2$. We set $S_2 = \{\{v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (\overline{v^{[2n-1]}})^2, v^{[2n-1]})^3, (\overline{v^{[2n-1]}})^3, ....(v^{[2n-1]})^{n-1}, ((v^{[2n-1]})^{n-1})^1\}\}$.

Suppose that $S = S_1$ when $(2n-1) \equiv 0 \pmod L$ ($S = S_1 \cup S_2$, , otherwise). Obviously, the subgraph induced by each element in $S$ is isomorphic to $P_L$ and $|S| = \left\lceil \frac{n-1}{L} \right\rceil$. Let $W(S) = \bigcup_{f \in s} f$. Since $AQ_n - W(S)$ is disconnected and one component of it is $\{u\}$, $k(AQ_n, P_L) \le \left\lceil \frac{2n-1}{L} \right\rceil$ and $k^s(AQ_n, P_L) \le \left\lceil \frac{2n-1}{L} \right\rceil$.

**Lemma 8.** *For* $n \ge 3$ *and* $4 \le L \le 2n-1$, $k^s(AQ_n, P_L) \ge \left\lceil \frac{2n-1}{L} \right\rceil$.

*Proof.* Let $F_n^*$ be a set of connected subgraphs in $AQ_n$, every element in the set is isomorphic to a connected subgraph of $P_L$ and $|F_n| \le \left\lceil \frac{2n-1}{L} \right\rceil - 1$. Thus $|V(F_n^*)| \le L \times (\left\lceil \frac{2n-1}{L} \right\rceil - 1) < 2n-1$. Since $k(AQ_n) = 2n-1$, $AQ_n - F_n^*$ is connected. Hence, the lemma holds.

By Lemma 7 and Lemma 8, we have the following theorem.

**Theorem 5.** *For* $n \ge 3$ *and and* $4 \le L \le 2n-1$, $k(AQ_n, P_L) = k^s(AQ_n, P_L) = \left\lceil \frac{2n-1}{L} \right\rceil$.

### 3.3 $k(AQ_n, C_N)$ and $k^s(AQ_n, C_N)$

At first, we discuss $k^s(AQ_n, C_N)$. Then we discuss $k(AQ_n, C_N)$.

#### 3.3.1 $k^s(AQ_n, C_N)$ with $3 \le N \le 2n-1$

Since $P_N$ is a connected subgraph of $C_N$, we have the following lemmas.

**Lemma 9.** *For* $n \ge 3$ *and* $3 \le N \le 2n-1$, $k^s(AQ_n, C_N) \le \left\lceil \frac{2n-1}{N} \right\rceil$.

**Lemma 10.** *For* $n \ge 3$ *and* $3 \le N \le 2n-1$, $k^s(AQ_n, C_N) \ge \left\lceil \frac{2n-1}{N} \right\rceil$.

*Proof.* Let $F_n^*$ be a set of connected subgraphs in $AQ_n$, every element in the set is isomorphic to a connected subgraph of $C_N$ with $|F_n^*| \le \left\lceil \frac{2n-1}{N} \right\rceil - 1$. Thus $V(F_n^*) \le N \times \left\lceil \frac{2n-1}{N} \right\rceil - 1 < 2n-1$. Since $k(AQ_n) = 2n-1$, $AQ_n - F_n^*$ is connected. Hence, the lemma holds.

By Lemma 9 and Lemma 10, we have the following theorem.

**Theorem 6.** *For* $n \ge 3$ *and* $3 \le N \le 2n-1$, $k^s(AQ_n, C_N) = \left\lceil \frac{2n-1}{N} \right\rceil$.

Now, we discuss $k(AQ_n, C_3)$ and $k(AQ_n, C_N)$ with $4 \le N \le 2n-1$.

### 3.3.2 $k(AQ_n, C_3)$

We have the following lemma.

**Lemma 11.** *For* $n \geq 6$, $k(AQ_n, C_3) \leq n-1$.

*Proof.* Let $u$ be an any vertex in $AQ_n$. We set $S_1 = \{\{u^1, u^2, \overline{u}^2\}\}$ and $S_2 = \{\{u^i, \overline{u}^i, (u^i)^{i-1}\} | 3 \leq i \leq n\}$.

Obviously, the subgraph induced by the element in $S_1$ is isomorphic to $C_3$. For $3 \leq i \leq n$, $\{\{u^i, \overline{u}^i\}, \{\{\overline{u}^i, (u^i)^{i-1}\}, \{(u^i)^{i-1}, u^i\}\}, \subseteq E(AQ_n)$. Thus, the subgraph induced by each element in $S = S_1 \cup S_2$ is isomorphic to $C_3$ and $|S| = n-1$. Let $W(S) = \bigcup_{f \in s} f$. Since $AQ_n - W(S)$ is disconnected and one component of it is $\{u\}$, $k(AQ_n, C_3) \leq n-1$.

**Lemma 12.** *Let* $F_n$ *be a* $C_3$*-structure set of* $AQ_n$ *with* $n \geq 6$. *If there exists an isolated vertex in* $AQ_n - V(F_n)$, *then* $|F_n| \geq n-1$.

*Proof.* Let $u$ be an any vertex in $AQ_n$. We set $W = \{x(x,u)\ \textit{is a hypercube edge}, x \in V(G)\}$ and $Z = \{y(y,u)\ \textit{is a complement edge}, y \in V(G)\}$. Clearly, $|W| = n$ and $|Z| = n-1$. By Property 1 and Property 2, each element in $F_n$ contains at most three distinct vertices in $N(u)$, namely, $u^i, \overline{u}^{i+1}$ and $u^{i+1}$ with $1 \leq i \leq n-1$ and $\{(\overline{u}^i, u^{i+1}), (\overline{u}^i, \overline{u}^{i+1}), (u^{i+1}, \overline{u}^{i+1})\} \subseteq E(AQ_n)$. Thus, the subgraph induced by $\{\overline{u}^i, u^{i+1}, \overline{u}^{i+1}\}$ is isomorphic to $C_3$. We set $B = \{b_i | b_i \in F_n$ and $\{\overline{u}^i, u^{i+1}, \overline{u}^{i+1}\} \subseteq (b_i) \cap N(u)\}$. Since each $V(b_i)$ contains two vertices in $Z$ and one vertex in $W$, $|B| \geq \left\lfloor \frac{n-1}{3} \right\rfloor$. In the following, we distinguish cases for the value of $|B|$.

**Case 1.** $|B| = 0$. Since each element in $F_n$ contains at most two distinct vertices in $N(u)$, $F_n \geq \left\lceil \frac{2n-1}{2} \right\rceil$.

**Case 2.** $|B| = 1$. Suppose that $\{\overline{u}^i, u^{i+1}, \overline{u}^{i+1}\} \subseteq V(b_i)$ with $1 \leq i \leq n-1$. Since each element in $F_n - B$ contains at most two distinct vertices in $N(u) - V(B)$, $|F_n| \geq 1 + \left\lceil \frac{2n-4}{2} \right\rceil = n-1$.

**Case 3.** $|B| = 2$. Suppose that $\{\overline{u}^i, u^{i+1}, \overline{u}^{i+1}\} \subseteq V(b_i)$, $\{\overline{u}^j, u^{j+1}, \overline{u}^{j+1}\} \subseteq V(b_j)$ with $1 \leq i \leq i, j < n-1$, and $|i-j| \geq 2$. Without loss of generality, we set $j > i$.

**Case 3.1.** $j - i = 2$. Then $\{\overline{u}^j, u^{j+1}, \overline{u}^{j+1}\} = \{\overline{u}^{i+2}, u^{i+3}, \overline{u}^{i+3}\}$. According to the definition of $AQ_n$, vertex $u^{i+2}$ is not adjacent to the vertices in $N(u) - V(B)$. As a result, there is an element $a \in F_n - B$ such that $u^{i+2} \in V(a)$ and $V(a) \cap \{N(u) - V(B)\} = 1$. Since the element in $F - B - \{a\}$ contains at most two distinct vertices in $N(u) - V(B) - V(a)$, $|F_n| \geq 2 + 1 + \left\lceil \frac{2n-8}{2} \right\rceil = n-1$.

**Case 3.2.** $j - i = 3$. Then $\{\overline{u}^j, u^{j+1}, \overline{u}^{j+1}\} = \{\overline{u}^{i+3}, u^{i+4}, \overline{u}^{i+4}\}$. In the following, we distinguish cases for the number of elements containing three vertices $u^{i+2}, \overline{u}^{i+2}$, and $u^{i+3}$, in $F_n - B$. We use $S$ denote the number of elements further deal with the following cases.

**Case 3.2.1.** $S = 3$. Then there are three distinct elements in $N(u) - V(B)$ that contain one of three vertices $u^{i+2}, \overline{u}^{i+2}$, and $u^{i+3}$, respectively. By the definition of $AQ_n$, vertex $u^{i+2}$ is not adjacent to the vertices in $N(u) - V(B) - \{u^{i+2}, u^{i+3}\}$. Then there is an element $a_1 \in F_n - B$ such that $u^{i+1} \in V(a_1)$ and $V(a_1) \cap \{N(u) - V(B) - \{\overline{u}^{i+2}, u^{i+3}\}\} = 1$. For the cases of vertices $\overline{u}^{i+2}$ and $u^{i+3}$, the discussions are similar to that of vertex $u^{i+2}$ and we set $\overline{u}^{i+2} \in a_2$, $u^{i+3} \in a_3$. Since each element in $F_N - B - \{a_1, a_2, a_3\}$ contains at most two distinct vertices in $N(u) - V(B) - V(a_1) - V(a_2) - V(a_3)$, $|F_n| \geq 2 + 3 + \left\lceil \frac{2n-10}{2} \right\rceil = n$.

**Case 3.2.2.** $S = 2$. Then there are two distinct elements in $N(u) - V(B)$, one of which contains one vertex in $u^{i+2}, \overline{u}^{i+2}$ and $u^{i+3}$, and the other contains the other two vertices. We assume that $a_1$ contains one vertex in $u^{i+2}, \overline{u}^{i+2}$ and $u^{i+3}$ and $a_2$ contains the other two vertices.

**Case 3.2.2.1.** $u^{i+2} \in V(a_1)$ and $\{u^{i+2}, \overline{u}^{i+3}\} \subseteq V(a_2)$. Similar to the discussion of Case 3.2.1, we have $V(a_1) \cap \{N(u) - V(B) - \{\overline{u}^{i+2}, u^{i+3}\}\} = 1$. Vertex $u^{i+2}$ or $u^{i+3}$ is not adjacent to the vertices in $N(u) - V(B) - \{u^{i+2}\}$ and $(u^{i+2}, u^{i+3}) \in E(AQ_n)$. Then $\{\overline{u}^{i+2}, u^{i+3}\} \subseteq a_2$ and $V(a_2) \cap \{N(u) - V(B) - \{\overline{u}^{i+2}\}\} = 2$. Since each element in $F_n - B - \{a_1, a_2\}$ contains at most two distinct vertices in $N(u) - V(B) - V(a_1) - V(a_2)$, $|F_n| \geq 2 + 1 + 1 + \left\lceil \frac{2n-10}{2} \right\rceil = n-1$. For the case of vertices $u^{i+3} \in V(a_1)$ and $\{u^{i+2}, u^{i+2}\} \not\subset V(a_2)$, the discussion is similar.

**Case 3.2.2.2.** $u^{i+2} \in V(a_1)$ and $\{u^{i+2}, u^{i+3}\} \not\subset V(a_2)$. Since $(u^{i+2}, u^{i+3})\} \in E(AQ_n)$, this situation does not exist.

**Case 3.2.3.** $S = 1$. Since $\{u^{i+2}, u^{i+3}\} \in E(AQ_n)$, this situation does not exist.

**Case 3.3.** $j-i\geq 4$. We set $U_{ij}=\{u^{i+2},\overline{u}^{i+2},...,u^{j-1}\}$. We will calculate the number of elements containing $U_{ij}$ in $F_n-B$. Clearly, $5\leq|U|\leq 2n-7$. Since each element contains at most two distinct vertices of $N(u)-V(B)$ in $F_n-B$ and $U\equiv 1$ (mod 2), similar to the discussion in Case 3.2.1, we have $|F_n|\geq 2+1+\left\lceil\frac{2n-8}{2}\right\rceil=n-1$.

**Case 4.** $|B|\geq 3$. If $b_i,b_j\in B$ and there is no $b_k\in B$ with $i<k<j$, we set $U_{ij}=\{u^{i+2},\overline{u}^{i+2},...,u^{j-1}\}$. According to the discussion of Case 3, the minimum number of elements that contain all vertices of $U_{ij}$ in $F_n-B$ is $\left\lceil\frac{U_{ij}}{2}\right\rceil$. Thus, $|F_n|\geq|B|+(|B|-1)+\left\lceil\frac{2n-1-3\times|B|-(|B|-1)}{2}\right\rceil=n-1$.

In summary, the lemma holds.

**Lemma 13.** *For* $n\geq 6$, $k(AQ_n,C_3)\geq n-1$.

*Proof.* We will prove this lemma by contradiction. Let $F_n^*$ be a $C_3$-structure set of $AQ_n$ and $|F_n^*|\leq n-2$. If $AQ_n-V(F_n^*)$. is disconnected, then we let $R$ be the smallest component of $AQ_n-V(F_n^*)$. Note that $|V(F_n^*)|\leq 3\times(n-2)=3n-6$. By Lemma 1, we have $3n-6<4n-8$ for $n\geq 6$. Hence $|V(R)|=1$. Furthermore, we assume that vertex $u\in V(R)$. By Lemma 12, $|N(u)\bigcap V(F_n^*)|\leq 2n-2<2n-1$, which means that there exists at least one neighbor of $u$ in $AQ_n-V(F_n^*)$. Therefore, we have $|V(R)|\geq 2$, a contradiction. Thus, $AQ_n-V(F_n^*)$ is connected. The lemma holds.

By Lemma 11 and Lemma 13, we have the following theorem.

**Theorem 7.** *For* $n\geq 6$, $k(AQ_n,C_3)=n-1$.

### 3.3.3 $k(AQ_n,C_3)$ with $4\leq N\leq 2N-1$

**Lemma 14.** *For* $\geq 6$ *and* $4\leq N\leq 2n-1$, $k(AQ_n,C_N)\leq\left\lceil\frac{2n-1}{N-1}\right\rceil$.

*Proof.* Let $u$ be an arbitrary vertex in $AQ_n$. According to the parity of $N$, we will discuss the following two cases.

**Case 1.** $N$ is odd. We set

$$S_1=\{\{v^{[(N-1)i+1]},v^{[(N-1)i+2]},...,v^{[(i+1)(N-1)]}\},(\overline{v^{[(i+1)(N-1)]}})^{\left\lfloor\frac{(N-1)i+1}{2}\right\rfloor+1}\}\,|\,0\leq i<\left\lfloor\frac{2n-1}{N-1}\right\rfloor\}.$$

**Case 1.1.** $(2n-1)\equiv 1$ (mod $(N-1)$).

**Case 1.1.1.** $N\leq 2n-3$.. We set

$$S_2=\{\{v^{[2n-1]},(v^{[2n-1]})^1,(v^{[2n-1]})^2,(\overline{v^{[2n-1]}})^2,...,(v^{[2n-1]})^{\left\lfloor\frac{N}{2}\right\rfloor},(\overline{v^{[2n-1]}})^{\left\lfloor\frac{N}{2}\right\rfloor},(v^{[2n-1]})^{\left\lfloor\frac{N}{2}\right\rfloor}\}\}.$$

**Case 1.1.2.** $N=2n-1$. We set

$$S_2=\{\{v^{[2n-1]},(v^{[2n-1]})^1,(v^{[2n-1]})^2,(\overline{v^{[2n-1]}})^2,...,(v^{[2n-1]})^{n-3},(\overline{v^{[2n-1]}})^{n-3},(v^{[2n-1]})^{n-3},((\overline{v^{[2n-1]}})^{n-2})^{n-4},\overline{((\overline{v^{[2n-1]}})^{n-2})^{n-4}},((\overline{v^{[2n-1]}})^{n-2})^{n-3},\overline{((\overline{v^{[2n-1]}})^{n-2})^{n-3}}\}\}.$$

**Case 1.2.** $(2n-1)\equiv 3$. (mod $(N-1)$). We set

$$S_2=\{\{v^{[2n-1]},v^{[2n-3]},v^{[2n-2]},(v^{[2n-2]})^{n-1-\frac{N-3}{2}},(\overline{v^{[2n-2]}})^{n-1-\frac{N-3}{2}},(v^{[2n-2]})^{n-1-\frac{N-3}{2}},...,(v^{[2n-2]})^{n-2},(\overline{v^{[2n-2]}})^{n-2}\}\}.$$

**Case 1.3.** $(2n-1)>3$. (mod $(N-1)$). We set

$$S_2=\{\{v^{[\left\lfloor\frac{2n-1}{N-1}(N-1)+1\right\rfloor]},v^{[\left\lfloor\frac{2n-1}{N-1}(N-1)+2\right\rfloor]},...,v^{[2n-3]},$$

$$v^{[2n-1]},v^{[2n-2]},(v^{[2n-2]})^{\left\lfloor\frac{\left\lfloor\frac{2n-1}{N-1}(N-1)+1\right\rfloor}{2}+2\right\rfloor-\frac{N-(2n-1\left\lfloor\frac{2n-1}{N-1}\right\rfloor(N-1))}{2}},$$

$$(\overline{v^{[2n-2]}})^{\left\lfloor\frac{\left\lfloor\frac{2n-1}{N-1}(N-1)+1\right\rfloor}{2}+2\right\rfloor-\frac{N-(2n-1-\left\lfloor\frac{2n-1}{N-1}\right\rfloor(N-1))}{2}},$$

$$(v^{[2n-2]})^{\left\lfloor\frac{\left\lfloor\frac{2n-1}{N-1}(N-1)+1\right\rfloor}{2}+3\right\rfloor-\frac{N-(2n-1-\left\lfloor\frac{2n-1}{N-1}\right\rfloor(N-1))}{2}},$$

$$(\overline{v^{[2n-2]}})^{\left\lfloor\frac{\left\lfloor\frac{2n-1}{N-1}(N-1)+1\right\rfloor}{2}+3\right\rfloor-\frac{N-(2n-1-\left\lfloor\frac{2n-1}{N-1}\right\rfloor(N-1))}{2}},...,$$

$$(\overline{v^{[2n-2]}})^{\frac{\left\lfloor\frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}}\}\}.$$

**Case 2.** $N$ is even. We set

If $0\leq i<\left\lfloor\frac{2n-1}{N-1}\right\rfloor$ and $i\equiv 0$ (mod 2) and $N=4$, then

$$S_1=\{v^{[3i+1]},v^{[3i+3]},v^{[3i+2]},(v^{[3i+1]})^{\left\lfloor\frac{3i+1}{2}\right\rfloor+3}\};$$

If $0\leq i<\left\lfloor\frac{2n-1}{N-1}\right\rfloor$ and $i\equiv 0$ (mod 2) and $N\neq 4$, then

$$S_1=\{\{v^{[(N-1)+1]},v^{[(N-1)+2]},...,v^{[(i+1)(N-1)+1]},(\overline{v^{[(i+1)(N-1)]}})^{\left\lfloor\frac{(N-1)i+1}{2}+1\right\rfloor}\}.$$

$$S_2=\{\{v^{[(N-1)+1]},v^{[(N-1)+2]},...,v^{[(i+1)(N-1)+1]},(v^{[(i+1)(N-1)+1]})^{\left\lfloor\frac{(N-1)i+1}{2}+1\right\rfloor}\}\,|\,0\leq i<\left\lfloor\frac{2n-1}{2}\right\rfloor\text{ and }i=1\text{ (mod 2)}\}.$$

If $(2n-1)\equiv 0$ (mod $(N-1)$), then $S_3=\phi$; otherwise, we will discuss the following several cases,

**Case 2.1.** $(2n-1)\equiv 1$ (mod $(N-1)$). We set

$$S_3=\{\{v^{[2n-1]},(v^{[2n-1]})^1,(v^{[2n-1]})^2,(\overline{v^{[2n-1]}})^2,(v^{[2n-1]})^3,(\overline{v^{[2n-1]}})^3,...,(v^{[2n-1]})^{\frac{N}{2}},(\overline{v^{[2n-1]}})^{\frac{N}{2}}\}\}.$$

**Case 2.2.** $(2n-1)\equiv 2.$ (mod $(N-1)$).

**Case 2.2.1.** $N<n.$ We set

$$S_3=\{\{v^{[2n-1]},(v^{[2n-2]}),(v^{[2n-2]})^{n-1},((v^{[2n-2]})^{n-1})^{n-2},\ldots,(((v^{[2n-2]})^{n-1})\ldots)^{n-(N-2)}\}\}.$$

**Case 2.2.2.** $N=2n-2.$ We set

$$S_3=\{\{v^{[2n-1]},(v^{[2n-2]}),(v^{[2n-2]})^1,(v^{[2n-2]})^2,(\overline{(v^{[2n-2]})})^2,(v^{[2n-2]})^3,(\overline{(v^{[2n-2]})})^3,\ldots,(v^{[2n-2]})^{n-2},(\overline{(v^{[2n-2]})})^{n-2},(v^{[2n-2]})^{n-1}\}\}.$$

**Case 2.3.** $(2n-1)\equiv 3.$ (mod $(N-1)$). We set

$$S_3=\{\{v^{[2n-2]},(v^{[2n-3]}),(v^{[2n-1]}),(\overline{v^{[2n-1]}})^{2+n-N},(\overline{v^{[2n-1]}})^{3+n-N},\ldots,(\overline{v^{[2n-1]}})^{n-2}\}\}.$$

**Case 2.4.** $(2n-1)\equiv 4.$ (mod $(N-1)$). We set

**Case 2.4.1.** $N<n.$

$$S_3=\{\{v^{[2n-4]},v^{[2n-3]},v^{[2n-1]},v^{[2n-2]},(v^{[2n-2]})^{4+n-N},(v^{[2n-2]})^{5+n-N},\ldots,(v^{[2n-2]})n-1\}\}.$$

**Case 2.4.2.** $N=2n-4.$ We set

$$S_3=\{\{v^{[2n-4]},v^{[2n-3]},v^{[2n-1]},v^{[2n-2]},(\overline{v^{[2n-2]}})^3,(v^{[2n-2]})^4,(\overline{v^{[2n-2]}})^4,(v^{[2n-2]})^5,(\overline{v^{[2n-2]}})^5,\ldots,(\overline{v^{[2n-2]}})^{n-2},(v^{[2n-2]})^{n-1}\}\}.$$

**Case 2.5.** $(2n-1)$ is odd (mod $(N-1)$)(except 1 and 3). We set

$$S_3=\{\{v^{[\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1]},v^{[\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+2]},\ldots,v^{[2n-3]},v^{[2n-1]},v^{[2n-2]},(\overline{v^{[2n-2]}})^{[\frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}-N+2n-\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1]},(\overline{v^{[2n-2]}})^{[\frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}-N+2n-\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+2]},\ldots,(\overline{v^{[2n-2]}})^{[\frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}+1}\}\}.$$

**Case 2.6.** $(2n-1)$ is even(mod $(N-1)$)(except 0, 2 and 4).

**Case 2.6.1.** $N-n.$. We set

$$S_3=\{\{v^{[\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1]},v^{[\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+2]},\ldots,v^{[2n-3]},v^{[2n-1]},v^{[2n-2]},(\overline{v^{[2n-2]}})^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}+1\right\rfloor},$$

$$\overline{((\overline{v^{[2n-2]}})^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}+1\right\rfloor})}^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}+1\right\rfloor-N+2n-\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1},$$

$$\overline{((\overline{v^{[2n-2]}})^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}+1\right\rfloor})}^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}\right\rfloor-N+2n-\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+2},\ldots,$$

$$\overline{((\overline{v^{[2n-2]}})^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}+1\right\rfloor})}^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}\right\rfloor}\}\}.$$

**Case 2.6.2.** $n\le N\le 2n-6.$

**Case 2.6.2.1.** $n=N-1.$ We set

$$S_3=\{\{v^{[\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1]},v^{[\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+2]},\ldots,v^{[2n-3]},v^{[2n-1]},v^{[2n-2]},(\overline{v^{[2n-2]}})^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}+1\right\rfloor},(v^{[2n-2]})^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}+1\right\rfloor}\}\}.$$

**Case 2.6.2.2.** $n\ne N-1$ We set

$$S_3=\{\{v^{[\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1]},v^{[\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+2]},\ldots,v^{[2n-1]},v^{[2n-2]},(\overline{v^{[2n-2]}})^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}+1\right\rfloor},$$

$$\overline{((\overline{v^{[2n-2]}})^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}+1\right\rfloor})}^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}\right\rfloor-N+n+1},$$

$$\overline{((\overline{v^{[2n-2]}})^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}+1\right\rfloor})}^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}\right\rfloor-N+n+1},$$

$$\overline{((\overline{v^{[2n-2]}})^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}+1\right\rfloor})}^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}\right\rfloor-N+n+2},$$

$$\overline{((\overline{v^{[2n-2]}})^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}+1\right\rfloor})}^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}\right\rfloor-N+n+2},\ldots,$$

$$\overline{((\overline{v^{[2n-2]}})^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}+1\right\rfloor})}^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}\right\rfloor-1},.$$

$$\overline{((\overline{v^{[2n-2]}})^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}+1\right\rfloor})}^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}\right\rfloor-1},.$$

$$(v^{[2n-2]})^{\left\lfloor \frac{\left\lfloor \frac{2n-1}{N-1}\right\rfloor(N-1)+1}{2}+1\right\rfloor}\}\}.$$

We set $S=S_1\bigcup S_2$ when $N$ is odd or $(2n-1)\equiv 0$ (mod $(N-1)$) ( $S=S_1\bigcup S_2\bigcup S_3$, otherwise). According to the definition of $AQ_n$, the subgraph induced by each element in $S$ is isomorphic to $C_N$ and $|S|=\left\lceil \frac{2n-1}{N-1}\right\rceil$. Let $W(S)=\bigcup_{f\in s} f$. Since $AQ_n-W(S)$ is disconnected and one component of it is $\{u\}$, $k(AQ_n,C_N)\le \left\lceil \frac{2n-1}{N-1}\right\rceil$. Figure 5 shows a $C_5$-structure-cut in $AQ_6$.

**Lemma 15.** *Let $F_N$ be a $C_N$-structure set of $AQ_n$ with $n\ge$ and $4\le N\le 2n-1$. If there exists an isolated vertex in $AQ_n-V(F_n)$, then $|F_n|\ge\left\lceil \frac{2n-1}{N-1}\right\rceil$.*

*Proof.* Let $u$ be an any vertex in $AQ_n$. According to the definition of $AQ_n$, Property 1 and Property 2, any $N$ neighbors of $u$ cannot form a $C_N$ and each element in $F$ contains at most $N-1$ distinct vertices in $N(u)$. Thus, $|F_n|\ge\left\lceil \frac{2n-1}{N-1}\right\rceil$.

**Lemma 16.** *For $n\ge 6$ and $4\le N\le 2n-1$,* $k(AQ_n,C_N)\ge \left\lceil \frac{2n-1}{N-1}\right\rceil$.

*Proof.* We will prove this lemma by contradiction. Let $F_n^*$ be a $C_N$ -structure set of $AQ_n$ and $|F_n^*| \leq 1\left\lceil \frac{2n-1}{N-1} \right\rceil -1$. If $AQ_n - V(F_n^*)$ is disconnected, then we let $R$ be the smallest component of $AQ_n - V(F_n^*)$. Not that $|V(F_n^*)| \leq N \times 1\left\lceil \frac{2n-1}{N-1} \right\rceil -1)$. By Lemma 1, we have $N \times 1\left\lceil \frac{2n-1}{N-1} \right\rceil -1 < 4n-8$ for $n \geq 6$. Hence $|V(R)| = 1.$ . Furthermore, we assume that vertex $u \in V(R)$. By Lemma 15, $|N(u) - V(F_n^*) \leq 2n-2 < 2n-1$, which means that there exists at least one neighbor of $u$ in $AQ_n - V(F_n^*)$. Therefore, we have $|V(R)| \geq 2$, a contradiction. Thus, $AQ_n - V(F_n^*)$ is connected. The lemma holds.

By Lemma 14 and Lemma 16, we have the following theorem.

**Theorem 8.** *For* $n \geq 6$ *and* $4 \leq N \leq 2n-1$, $k(AQ_n, C_N) = \left\lceil \frac{2n-1}{N-1} \right\rceil$.

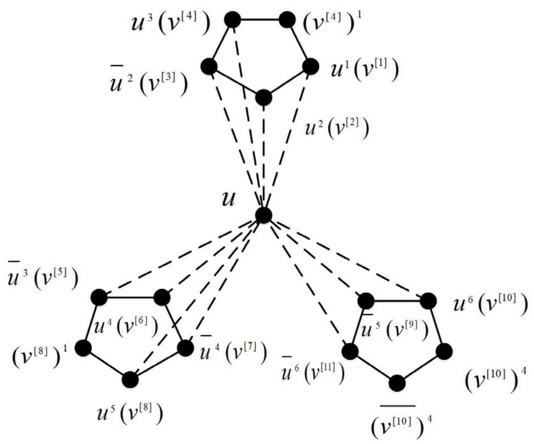



**Figure 5.** A $C_5$ -structure-cut in $AQ_6$

## 4 Conclusion

In this paper, we study the *H*-structure and *H*-substructure connectivity of augmented cube. The results are summarized as follows.

(1) $k(AQ_n, K_{1,M}) = k^s(AQ_n, K_{1,M}) =$

$$\begin{cases} 2n-1 & \text{for } n \geq 4 \text{ and } K_{1,M} = K_1, \\ \left\lceil \frac{2n-1}{1+M} \right\rceil & \text{for } n \geq 4 \text{ and } 1 \leq M \leq 3, \\ \left\lceil \frac{n-1}{2} \right\rceil & \text{for } n \geq 6 \text{ and } 4 \leq M \leq 6. \end{cases}$$

(2) $k(AQ_n, P_L) = k^s(AQ_n, P_L) = \left\lceil \frac{2n-1}{L} \right\rceil$ *for* $n \geq 3$ *and* $1 \leq L \leq 2n-1$.

(3) $k^s(AQ_n, C_N) = \left\lceil \frac{2n-1}{L} \right\rceil$ *for* $n \geq 3$ *and* $3 \leq N \leq 2n-1$.

(4) $k^s(AQ_n, C_N) = \begin{cases} n-1 & \text{for } n \geq 6 \text{ and } N = 3, \\ \left\lceil \frac{2n-1}{N-1} \right\rceil & \text{for } n \geq 6 \text{ and } 4 \leq n \leq 2n-1. \end{cases}$

We study $K_{1,M}$ when $M$ is small $(1 \leq M \leq 6)$ in this paper. In the case of ensuring reliable communication, the number of faulty vertices that can be tolerated in augmented cube is almost twice the traditional connectivity under ideal conditions. In the future, we may focus on $M \geq 7$, which will make us more comprehensively realize the fault-tolerant ability of augmented cube. On the other hand, we may also study other properties of augmented cube with structure faults such as hamiltonian properties [30-31].

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