Random Walks on the Folded Hypercube

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Abstract

Random walks are basic mechanism for many dynamic processes on the network. In this paper, we study the global mean first-passage time (GMFPT) of random walks on the n-dimensional folded hypercube FQ_n. FQ_n is a variation of the hypercube Q_n by adding complementary edges, and characterized with the superiorities of smaller diameter and higher connectivity than the hypercube. We initiate a more concise formula to the Kirchhoff index by using the spectra of the Laplace matrix of FQ_n. We also obtain the explicit formula to GMFPT, and the exponent of scaling efficiency characterizing the random walks is further determined, finding that it takes less time when random walks on FQ_n than on Q_n. Moreover, we explore random walks on the FQ_n considering a given trap. Finally, we make some comparison with Q_n in Kirchhoff index, noticing a more effective traffic on FQ_n.

Keywords: Random walks, Folded hypercube, Mean first-passage time, Kirchhoff index

1 Introduction

Random walks are irregular forms of variation, and each step in the process of change is random, have appealed lots of interest, ranging back from as early as 1905, to nowadays [2-3]. In general, random walks are assumed to have the memoryless nature of Markov chains [5]. That is, each state value depends on the previous finite state. Random walks theory have a wide range of applications in optimization [6], machine learning [7], engineering [8], artificial intelligence [9] and other fields [10-11, 16, 29]. Some properties of random walks, such as dispersal distributions [12], first-passage times (sometimes named hitting time) [4], and encounter rates [13], have been studied intensively.

Specially, to have a further understanding of the characteristics of graph, many studies have been carried out on the first-passage time of random walks (FPT). The first-passage time refers to the time when a given node i first reaches or exceeds another given node j, and thus derived to mean FPT (MFPT, which is an average value over all starting sites), global MFPT (GMFPT, the mean FPT for whole pairs of nodes). Many literatures have discussed the MFPT on generalized graphs. Tejedor et al. [14] posed a general framework to give a minimal scaling on the GMFPT to a target site on complex networks. Condami et al. [4] revealed an explicit scaling dependence of the MFPT on the volume of the constrained domain and the source-target distance, using probability distribution. Elsässer and Sauerwald [19] took multiple random walks into account and proposed the tight bounds for the cover time. Meanwhile, others focused on specific graphs and gave the MFPT scaling. By a large number of numerical calculations, Zhang et al. [15] showed the explicit expression solution for GMFPT on hypercubes, with the GMFPT scaling $\langle F \rangle \sim V_n$ and Zhang et al. [17] obtained an exact expression of the MFPT random walks on the T-graph, with the GMFPT scaling $\langle F \rangle \sim V_n^{\frac{\ln 6}{\ln 3}}$. Combining with the recursive properties, Comellas et al. [18] recovered the MFPT for random walks on recursive trees, with the GMFPT scaling $\langle F \rangle \sim V_n \log(V_n)$. Zhang et al. [20] also gave a solution for the MFPT for random walks on pseudofractal scale-free web, with the GMFPT scaling $\langle F \rangle \sim V_n^{\frac{\ln 2}{\ln 3}}$. These are quantitative indicators pointing to transport efficiency in a graph.

Obviously, numerous efforts have been made in exploring the characteristics on the FPT of graphs. Still, few works focus on FPT of the n-dimensional folded hypercube FQ_n, which was proposed by El-Amawy and Latifi [21] and constitute a variation of the hypercube networks with a large numbers of appealing properties, see [22] for its strong and conditional diagnosability, see [23] for its path embedding, see [24] for its connectivity, etc. All these superiorities make FQ_n widely used in parallel computing systems.

In this paper, we provide an explicit expression to solve the GMFPT by deriving through the spectra of *Laplacian matrix* of the folded hypercube, noting that though FQ_n has more edges than Q_n, it takes almost just 2^n units of time when random walks take place. At

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the same time, we also present a formula to K_f, which is different from the precious [28] and more comprehensive. The results indicate that the K_f in FQ_n is smaller than in Q_n, which means a more effective traffic in FQ_n. Moreover, considering a node with trap, we explore the time walking randomly over all notes except the trap.

The rest of this paper is organized as follows: In Section 2, we propose some terms and notations may used throughout the whole paper; In Section 3, we give the exact solutions to random walks on folded hypercubes and explore the mean first-passage time (MFPT) with a located trap; In Section 4, we have some discussion in terms of K_f and GMFPT when compared with Q_n. Finally, we give the conclusions.

2 Preliminaries

In this section, we first present some terms and notations used throughout the paper. We then review the hypercube and folded hypercube.

2.1 Terminology and Notation

Let $G = (V, E)$ be a graph, where the vertex set V is a nonempty and finite set, and the edge set E is a subset of $\{(i, j) \mid (i, j)\ \textit{is an unordered pair of}\ V\}$.

The mean first-passage time [1] is portraying as follows:

$$F_{ij} + F_{ji} = 2E \times r_{ij}, \quad \textbf{(1)}$$

where F_{ij} denotes the FPT for the walker walking from i to j, E denotes the whole number of edges of G, and r_{ij} is introduced by Klein and Randié [25] to depict the effective resistance between nodes i and j in G. Moreover, they introduced the Kirchhoff index (K_f) to capture the sum of resistance distances of all pairs of nodes, i.e.,

$$K_f = \sum_{i<j} r_{ij} . \quad \textbf{(2)}$$

Since r_{ij} is not an easily quantifiable data, and *Laplacian matrix* can better reflect the nature of the network, Zhu et al. [26] showed another relationship as follows:

$$K_f = V_n \times \sum_{i=2}^{V_n} \frac{1}{\lambda_i}, \quad \textbf{(3)}$$

where V_n is the number of nodes ($V_n \geq 2$, and $0 = \lambda_1 \leqslant \lambda_2 \leqslant ,\ldots, \leqslant \lambda_{V_n}$ are the eigenvalues of *Laplacian matrix* L, where $L = D - A$). Let D be the diagonal matrix whose *ith* diagonal entry d_i is the degree of the vertex V_i($1 \leq i \leq n$), while A denotes Adjacency matrix, and L defined as follows:

$$L_{ij} = \begin{cases} d_i, & i = j, \\ -1, & i \neq j, \textit{and} \quad \textit{edge} \quad (i, j) \quad \textit{belongs} \quad \textit{to} \quad G, \\ 0, & \textit{otherwise}. \end{cases}$$

Concurrently, we denote $\overline{F_{ij}}$ to be the expected time for walking by random from nodes i to j, and $\langle F \rangle_n$ are the average of expected time over all node pairs. The latter, is also named as global mean first passage time (GMFPT). $F_{sum}(n)$ denotes the sum of $\overline{F_{ij}}$ over whole pairs of nodes.

2.2 Folded Hypercubes

The following we will introduce some notations about hypercubes and folded hypercubes. The topology of a hypercube network is an n -dimensional hypercube, which is an undirected graph, denoted as Q_n . It can be defined by the following sequence,

$$V = \{x_1 x_2, ..., x_n : x_i \in 0, 1, i = 1, 2, \cdots, n\} .$$

The two nodes in Q_n are connected by edges if and only if there exists one and only one different coordinate. That means, for node $x = x_1 x_2 ... x_n$ and node $y = y_1 y_2 ... y_n$, always have $\sum_{i=1}^{n} |x_i - y_i| = 1$.

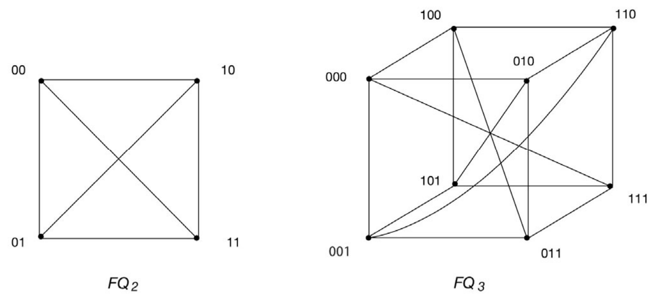

Figure 1. Two-dimensional folded hypercube FQ_2 and three-dimensional folded hypercube FQ_3

The n -dimensional folded hypercube FQ_n [24] is obtained by adding complementary edges to the hypercube Q_n (see *Figure 1*). Clearly, FQ_n is $(n+1)$ -regular and $(n+1)$ -connected. Let V_n and E_n be the total number of nodes and edges in FQ_n, then $V_n = 2^n$, $E_n = (n+1)2^{(n-1)}$. The diameter of n-dimensional FQ_n is $\left\lceil \frac{n}{2} \right\rceil$, and the folded hypercube preserve the symmetric characteristic of hypercubes. The unique superiorities make the folded hypercube considered to be a challenging and attractive network structure to replace the hypercube network. It is also a preferred structure in parallel processing and parallel computing systems.

Lemma1. [24] Let S_p be a spectra of *Laplacian matrix of* FQ_n , then

$$S_p = \begin{bmatrix} 0 & 4 & 8 & \cdots & 4\left\lceil \frac{n}{2} \right\rceil \\ C_{n+1}^0 & C_{n+1}^2 & C_{n+1}^4 & \cdots & C_{n+1}^{2\left\lfloor \frac{n}{2} \right\rfloor} \end{bmatrix}.$$

3 A rigorous Solution for GMFPT

In this section, by mathematical deducing, we initiate an exact solution to GMFPT when random walks in FQ_n. Further, since the expression of GMFPT is constituted of polynomials, we determine the scaling of GMFPT to simplify the calculation. Meanwhile, we discuss the GMFPT with a trap.

3.1 GMFPT Over all Node Pairs

It is well known that $\langle F \rangle_n$ (GMFPT over all node pairs) and F_{sum} (the sum of $\overline{F_{ij}}$ over all note pairs) can be shown as the following expression

$$\langle F \rangle_n = \frac{F_{sum}(n)}{V_n(V_n-1)} = \frac{1}{V_n(V_n-1)} \sum_{i \neq j} \sum_{j}^{V_n-1} \overline{F_{ij}}(n) \qquad \textbf{(4)}$$

and

$$F_{sum} = \sum_{i \neq j} \sum_{j}^{V_n-1} \overline{F_{ij}}(n) \,. \qquad \textbf{(5)}$$

According to the *Eqs*.(1)(2) showed above, we can obtain the results below, i.e.

$$F_{sum} = E_n \sum_{i \neq j} \sum_{j}^{V_n-1} r_{ij}(n) = 2E_n K_f(FQ_n) \,. \qquad \textbf{(6)}$$

Further, applying *Eqs*.(3)(6) , then we can get that

$$\begin{aligned} \langle F \rangle_n &= \frac{F_{sum}(n)}{V_n(V_n-1)} \\ &= \frac{1}{V_n(V_n-1)} \sum_{i \neq j} \sum_{j}^{V_n-1} \overline{F_{ij}}(n) \\ &= \frac{2E_n}{V_n(V_n-1)} V_n \sum_{i=1}^{V_n} \frac{1}{\lambda_i}. \end{aligned} \qquad \textbf{(7)}$$

Next, for convenience, we define $\Omega_n = \sum_{i=2}^{V_n} \frac{1}{\lambda_i}$ in *Eq*.(7) . Notice that Lemma 1 had informed us the spectra of *Laplacian matrix* of FQ_n as:

$$S_p = \begin{bmatrix} 0 & 4 & 8 & \cdots & 4\left\lceil \frac{n}{2} \right\rceil \\ C_{n+1}^0 & C_{n+1}^2 & C_{n+1}^4 & \cdots & C_{n+1}^{2\left\lfloor \frac{n}{2} \right\rfloor} \end{bmatrix}.$$

It follows that the eigenvalue of *Laplacian matrix* of FQ_n are $0, 4, 8, \ \ldots, 4\left\lceil \frac{k}{2} \right\rceil, \ldots, 4\left\lceil \frac{n}{2} \right\rceil$ and the multiplicity of $4\left\lceil \frac{k}{2} \right\rceil$ is $C_{n+1}^{2\left\lfloor \frac{k}{2} \right\rfloor}$. Then Lemma 2 will be useful as it recasts $\Omega_n = \sum_{i=2}^{V_n} \frac{1}{\lambda_i}$ as follows.

Lemma 2. The expression of Ω_n is

$$\Omega_n = \sum_{i=1}^{\left\lceil \frac{n}{2} \right\rceil} \frac{1}{4i} \binom{n+1}{2i} = \frac{1}{2} \sum_{i=1}^{\left\lceil \frac{n}{2} \right\rceil} \frac{1}{2i} \binom{n+1}{2i}.$$

Proof. Lemma 1 presents the spectra of *Laplacian matrix* of FQ_n, by deductive calculation, we can get the $S_p\{FQ_n\}$ and Ω_n of folded hypercubes with a lower dimension.

When $n=1, V_1=2$, then $S_p\{FQ_1\} = \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}$, since $\Omega_n = \sum_{i=2}^{V_n} \frac{1}{\lambda_i} = \frac{1}{4}$, it can also be recasted as: $\Omega_1 = \frac{1}{4} \times \binom{2}{2} = \frac{1}{4}$.

When $n=2, V_2=4$, then $S_p\{FQ_2\} = \begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}$, similarly, $\Omega_n = \sum_{i=2}^{V_n} \frac{1}{\lambda_i} = \frac{1}{4} \times 3 = \frac{3}{4}$, it can also be recasted as: $\Omega_2 = \frac{1}{4} \times \binom{3}{2} = \frac{3}{4}$.

When $n=3, V_3=8$, then $S_p\{FQ_3\} = \begin{pmatrix} 0 & 4 & 8 \\ 1 & 6 & 1 \end{pmatrix}$, similarly, $\Omega_n = \sum_{i=2}^{V_n} \frac{1}{\lambda_i} = \frac{1}{4} \times 6 + \frac{1}{8} \times 1 = \frac{13}{8}$,it can also be recasted as:

$$\Omega_3 = \frac{1}{4} \times \binom{4}{2} + \frac{1}{8} \times \binom{4}{4} = \frac{13}{8}.$$

When $n=4, V_4=16$, then

$$S_p\{FQ_4\} = \begin{pmatrix} 0 & 4 & 8 \\ 1 & 10 & 5 \end{pmatrix},$$

similarly, $\Omega_n = \sum_{i=2}^{V_n} \frac{1}{\lambda_i} = \frac{1}{4} \times 10 + \frac{1}{8} \times 15 = \frac{25}{8}$,

it can also be recasted as:

$$\Omega_4 = \frac{1}{4} \times \binom{5}{2} + \frac{1}{8} \times \binom{5}{4} = \frac{25}{8}.$$

When $n=5, V_5=32$, then

$$S_p\{FQ_5\}=\begin{pmatrix}0 & 4 & 8 & 12\\ 1 & 15 & 15 & 1\end{pmatrix},$$

similarly, $\Omega_n=\sum_{i=2}^{V_n}\frac{1}{\lambda_i}=\frac{1}{4}\times 15+\frac{1}{8}\times 15+\frac{1}{12}\times 1=\frac{137}{24}$,

it can also be recasted as:

$$\Omega_5=\frac{1}{4}\times\binom{6}{2}+\frac{1}{8}\times\binom{6}{4}+\frac{1}{12}\times\binom{6}{6}=\frac{137}{24}.$$

Subsequently, using mathematical induction, we can easily prove that the lemma conclusions mentioned above are all established.

Lemma 3. Let

$$a_n=\sum_{i=1}^{\left[\frac{n}{2}\right]}\frac{1}{2i}\binom{n+1}{2i}, \tag{8}$$

then

$$a_n=\frac{1}{2}\left(\frac{2^{n+1}}{n+1}+\frac{2^n}{n}+\cdots+\frac{2^2}{2}+\frac{2}{1}\right)-\left(\frac{1}{n+1}+\frac{1}{n}+\ldots+\frac{1}{2}+1\right).$$

Proof. First, we denote

$$b_n=\sum_{i=1}^{n}\binom{n}{i}\frac{1}{i}. \tag{9}$$

Because a_n and b_n have similar structure, we will make use of the existing results to proof the lemma. Here we introduce the following two identities referred to [27], i.e.,

$$H_n^{(r)}=1+\frac{1}{2^r}+\frac{1}{3^r}+\cdots+\frac{1}{n^r}$$

and

$$\sum_{i=1}^{n}\frac{x^i}{i}=H_n^{(1)}+\sum_{i=1}^{n}\frac{(x-1)^i}{i}\binom{n}{i}, \tag{10}$$

and we separately deal with the cases below.

Case 1. $x=2$.

We substitute it to $Eq.(10)$ and get that

$$\sum_{i=1}^{n}\frac{2^i}{i}=H_n^{(1)}+\sum_{i=1}^{n}\frac{1}{i}\binom{n}{i}. \tag{11}$$

Combining $Eq.(9)$ with $Eq.(11)$, we infer that

$$b_n=\left(\frac{2^n}{n}+\frac{2^{n-1}}{n-1}+\cdots+\frac{2^2}{2}+\frac{2}{1}\right)-\left(\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{2}+1\right).$$

Further, to present the exact polynomials of a_n, we consider the following situation.

Case 2. $x=0$.

In this case, we can easily obtain that

$$0=H_n^{(1)}+\sum_{i=1}^{n}\frac{(-1)^i}{i}\binom{n}{i}, \tag{12}$$

i.e.

$$0=\frac{-1}{1}\binom{n}{1}+\frac{1}{2}\binom{n}{2}-\frac{1}{3}\binom{n}{3}+\frac{1}{4}\binom{n}{4}+\cdots+\frac{(-1)^n}{n}\binom{n}{n}+1+\frac{1}{2}+\cdots+\frac{1}{n}.$$

Note that $x=1$ is useless during all the deriving, so we ignore it here. Combining $Eq.(11)$ with $Eq.(12)$, we get that

$$\sum_{i=1}^{n}\frac{2^i}{i}=H_n^{(1)}+\sum_{i=1}^{n}\frac{1}{i}\binom{n}{i}+H_n^{(1)}+\sum_{i=1}^{n}\frac{(-1)^i}{i}\binom{n}{i}.$$

Further, we can get a simplification as follows:

$$\sum_{i=1}^{n}\frac{1}{i}\binom{n}{i}+\sum_{i=1}^{n}\frac{(-1)^i}{i}\binom{n}{i}=\sum_{i=1}^{n}\frac{2^i}{i}-2H_n^{(1)}. \tag{13}$$

Specially, when $n=1$, both sides of formula come to 0. While $n\geq 2$, we can find that the odd terms on the left side of the equation are eliminated and the even terms are doubled. To get the expression of the sum of even terms, we recast the $Eq.(13)$ into

$$\sum_{i=1}^{\left[\frac{n-1}{2}\right]}\frac{1}{2i}\binom{n}{2i}=\frac{1}{2}\left(\sum_{i=1}^{n}\frac{2^i}{i}-2H_n\right), n\geq 2.$$

Thus, a_n can be expressed similarly, that is

$$\begin{aligned}a_n&=\sum_{i=1}^{\left[\frac{n}{2}\right]}\frac{1}{2i}\binom{n+1}{2i}\\&=\frac{1}{2}\left(\sum_{i=1}^{n+1}\frac{2^i}{i}-2H_{n+1}\right)\\&=\frac{1}{2}\left(\frac{2^{n+1}}{n+1}+\frac{2^n}{n}+\cdots+\frac{2^2}{2}+\frac{2}{1}\right)-\left(\frac{1}{n+1}+\frac{1}{n}+\cdots+1\right).\end{aligned} \tag{14}$$

Hence, Lemma 3 is proved completely.

Combining Lemma 2 with Lemma 3, we have the following theorem.

Theorem 1. The GMFPT of FQ_n is

$$\langle F\rangle_n=\frac{(n+1)2^{n-1}}{2^n-1}\left[\frac{1}{2}\left(\frac{2^{n+1}}{n+1}+\frac{2^n}{n}+\cdots+\frac{2^2}{2}+\frac{2}{1}\right)-\left(\frac{1}{n+1}+\frac{1}{n}+\cdots+\frac{1}{2}+1\right)\right].$$

Proof. Since

$$\begin{aligned}\Omega_n &= \frac{1}{2}a_n \\ &= \frac{1}{2}\left[\frac{1}{2}\left(\frac{2^{n+1}}{n+1}+\frac{2^n}{n}+\cdots\frac{2^2}{2}+\frac{2}{1}\right)-\left(\frac{1}{n+1}+\frac{1}{n}+\cdots+\frac{1}{2}+1\right)\right],\end{aligned} \tag{15}$$

we substitute $Eq.(15)$ into $Eq.(7)$, obtaining that

$$\begin{aligned}\langle F\rangle_n &= \frac{2E_n}{V_n-1}\Omega_n \\ &= \frac{(n+1)2^{n-1}}{2^n-1}\left[\frac{1}{2}\left(\frac{2^{n+1}}{n+1}+\frac{2^n}{n}+\cdots+\frac{2^2}{2}+\frac{2}{1}\right)-\left(\frac{1}{n+1}+\frac{1}{n}+\cdots+\frac{1}{2}+1\right)\right].\end{aligned}$$

Hence, Theorem 1 holds. □

Lemma 4. [15] For non-negative integers n, then $\lim_{n\to\infty}\sum_{n+1}=2$, where $\sum_{n+1}=\sum_{i=0}^{n}\frac{n+1}{2^i(n+1-i)}$.

Since the exact formula of $\langle F\rangle_n$ is some complicated when random walks on FQ_n as it is given in Theorem 1, we will evaluate the scaling of it to make calculation easier as follows.

Theorem 2. $\langle F\rangle_n \sim 2^n = V_n$.

Proof. By Theorem 1 and Lemma 4, we can deduce that

$$\begin{aligned}&\lim_{n\to\infty}\langle F\rangle_n \\ &= \lim_{n\to\infty}\frac{(n+1)2^{n-1}}{2^n-1}\left[\frac{1}{2}\left(\frac{2^{n+1}}{n+1}+\frac{2^n}{n}+\cdots+\frac{2^2}{2}+\frac{2}{1}\right)\right. \\ &\qquad\left.-\left(\frac{1}{n+1}+\frac{1}{n}+\cdots+\frac{1}{2}+1\right)\right] \\ &= \lim_{n\to\infty}\frac{2^n}{2^{n+1}-2}\left[\frac{1}{2}\left(2^{n+1}\sum_{i=0}^{n}\frac{n+1}{2^i(n+1-i)}\right)\right. \\ &\qquad\left.-(n+1)\left(\frac{1}{n+1}+\frac{1}{n}+\cdots+\frac{1}{2}+1\right)\right] \\ &= \lim_{n\to\infty}\frac{2^{n-1}\left[2^{n+1}2-(n+1)\left(\frac{1}{n+1}+\frac{1}{n}+\cdots+\frac{1}{2}+1\right)\right]}{2^{n+1}-2} \\ &= 2^n.\end{aligned}$$

Hence, as n approaches infinity, $\langle F\rangle_n$ obey the exponential function of 2^n, which can be expressed as $\langle F\rangle_n \sim 2^n$. As we can see, the value roughly coincide with V_n, implying that it takes almost 2^n units times when random walks on FQ_n.

3.2 GMFPT with a Trap

In this section, we discuss Markovian random walks on FQ_n in case of a given trap, labeled as i_T. Denote $\langle F_{i_T}\rangle_n$ as the average F_{ij} over all nodes in FQ_n, excluding the trap i_T. Note node i and j, we define that the random walks are of nearest neighbor type if

$$p_{ij}>0, \text{ implies } i\sim j,$$

and a simple random walk on node i, is given by

$$p_{ij}=\begin{cases}\frac{1}{deg(i)}, & i\sim j,\\ 0, & \text{otherwise.}\end{cases}$$

Then, specifying the transition probabilities as $P_{ij}\,(P_{ij}=(p_{ij})_{(V_n-1)\times(V_n-1)})$, and F_i as the first time for a given node i to reach i_T, we can depict F obeying the equation:

$$F_i=\sum_j P_{ij}F_j+1,$$

obviously, $i\neq i_T$. It can also be expressed as the following formula, i.e.

$$\mathbf{F}=\mathbf{Me},$$

where $M_{(V_n-1)\times(V_n-1)}=(\mathbf{I}-P_{ij})^{-1}$, $\mathbf{I}$ denotes (V_n-1)-dimensional unit matrix, while $\mathbf{e}$ is the (V_n-1)-dimensional unit vector, respectively.

We can obtain $\mathbf{F}$ when n is low, however, the exponential growth of V_n makes it hard to deal with when n is larger. To overcome the computational problems brought by high-dimension, we introduce the following expression as supplement:

$$\begin{aligned}\langle F_{i_T}\rangle_n &= \frac{1}{(V_n-1)}\sum_{i\neq j}\overline{F_{ij}} \\ &= \frac{1}{(V_n-1)}\times\frac{1}{V_n}\times V_n\times\sum_{i\neq j}\overline{F_{ij}} \\ &= \frac{1}{(V_n-1)}\times\frac{1}{V_n}\sum_{i=0}^{V_n-1}\sum_{i\neq j}\overline{F_{ij}} \\ &= \langle F\rangle_n.\end{aligned}$$

The results notify us that random walk on FQ from a given trap to other nodes takes the same time as

GMFPT. Furthermore, we found that the location of trap has little effect on the scaling of the GMFPT for random walks on FQ_n for the reason that FQ_n preserves the symmetric character of hypercubes. Thus, according to the definition, it is straightforward to verify that $\langle F_{i_T} \rangle_n$ can also be regarded as the MFPT.

4 Kirchhoff Index

In the following, we first provide a more concise formula to the Kirchhoff index in FQ_n. Then, we will take Q_n and FQ_n as examples to discuss in terms of Kirchhoff index.

Based on the $Eq.(3)$ and $Eq.(15)$, we can obtain the exact expression of $K_f(FQ_n)$ as following:

$$
\begin{aligned}
K_f(FQ_n) &= V_n\Omega_n \\
&= 2^n \frac{1}{2}\left[\frac{1}{2}\left(\frac{2^{n+1}}{n+1}+\frac{2^n}{n}+\cdots\frac{2^2}{2}+\frac{2}{1}\right) - \left(\frac{1}{n+1}+\frac{1}{n}+\cdots+\frac{1}{2}+1\right)\right] \\
&= 2^{n-1}\left(\frac{1}{2}\sum_{k=0}^{n}\frac{2^{k+1}}{k+1} - \sum_{k=0}^{n}\frac{1}{k+1}\right).
\end{aligned}
$$

It is worthwhile to note that the conclusion of our $K_f(FQ_n)$ is much more concise than the one referred in [28].

The previous literature [15] had shown us that

$$
\begin{aligned}
K_f(Q_n) &= V_n\Omega_n \\
&= 2^{n-1}\left[\left(\frac{2^n}{n}+\frac{2^{n-1}}{n-1}+\cdots\frac{2^2}{2}+\frac{2}{1}\right) - \left(\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{2}+1\right)\right] \\
&= 2^{n-1}\left(\sum_{k=1}^{n}\frac{2^k}{k} - \sum_{k=1}^{n}\frac{1}{k}\right),
\end{aligned}
$$

and we have arranged the values of low dimension into the following Table 1.

Table 1. Kirchhoff index of Q_n and FQ_n from $n=1$ to $n=10$

n	$K_f(Q_n)$	$K_f(FQ_n)$
1	1	0.5
2	5	3
3	19.333	13
4	68.667	50
5	236.53	182.67
6	809.07	653.33
7	2779.3	2322.7
8	9638.6	8272
9	33816	29626
10	1.2×10^5	1.07×10^5

As is manifested above, we can easily find that the Kirchhoff index in FQ_n are much less than in Q_n in the case of same dimension. This may be because the diameter of FQ_n is $\left\lceil \frac{n}{2} \right\rceil$ and FQ_n is $(n+1)$- regular and $(n+1)$ - connected. All these superior characteristics may lead to a more efficient traffic.

Moreover, we sort out the values of GMFPT based on Theorem 1 and [15] into Table 2, varying from $n=1$ to $n=15$. Figure 2 is given to make an intuitive perception. It is clearly that the values are neatly yield to exponential distributions, as is proved in Theorem 2.

Further, we find that random walks on FQ_n always takes less units time than on Q_n.

Table 2. GMFPT of Q_n and FQ_n from $n=1$ to $n=15$

n	GMFPT in Q_n	GMFPT in FQ_n
1	1	1
2	3.33333	3
3	8.28571	7.43
4	18.3111	16.67
5	38.1505	35.35
6	77.0539	72.59
7	153.188	146.31
8	302.386	291.95
9	595.518	579.76
10	1173.04	1149.11
11	2313.71	2276.96
12	4571.44	4515.11
13	9047.78	8956.75
14	17935.2	17787.9
15	35599.7	35357.2

So far, FQ_n had been proved that it has the superiorities of smaller diameter and higher connectivity than Q_n , and preserves the symmetric characteristic of Q_n . Additionally, we claim that GMPFT on FQ_n is roughly as short as 2^n, less than the ones on Q_n. This explicit result appears to be appealing because it strengthens that FQ_n is considered to be a powerful and challenging network structure to replace the hypercube network.

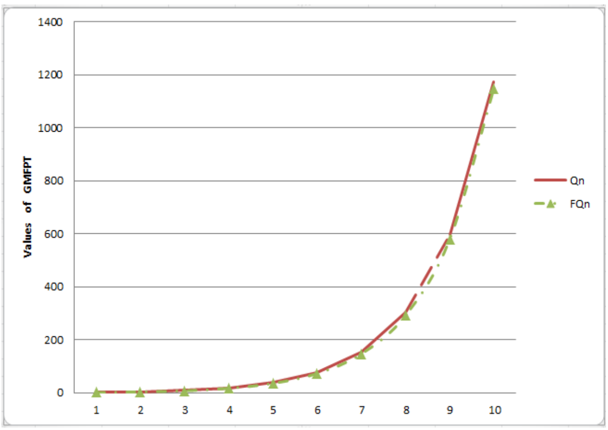

Figure 2. GMFPT of random walks on FQ_n and Q_n.

5 Conclusion

In this paper, we present a more concise formula to Kirchhoff index in terms of FQ_n by deriving using the spectra of *Laplacian matrix*. Concurrently, we give an exact expression to solve the GMFPT, and then determine the GMFPT scaling, which roughly equals to 2^n. Moreover, considering a node with trap, we find it takes the same time as GMFPT when walking randomly over all notes except the trap. In addition, an indication that the K_f in FQ_n is smaller than in Q_n indicates a more effective traffic in FQ_n. Furthermore, values of GMFPT in FQ_n is also smaller in quantity, confirming the analytical results again.

It makes sense to devote to the study of random walks on the FQ_n, for the reason that random walk is the basic mechanism for many dynamic processes on the network and FQ_n is a mature and appealing structure widely used. In fact, network based on the topology of FQ_n are extended to many areas, such as parallel computing systems, optimization, machine learning, engineering, artificial intelligence and other fields. We expect that by providing the explicit solution to GMFPT could guide and boost related studies of random walks, and lead to a more comprehensive understanding of the folded hypercube.

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References

[1] A. K. Chandra, P. Raghavan, W. L. Ruzzo, R. Smolensky, P. Tiwari, The Electrical Resistance of a Graph Captures Its Commute and Cover Times, *Computational Complexity*, Vol. 6, No. 4, pp. 312-340, December, 1996.

[2] F. Spitzer, *Principles of Random Walk*, Springer Science & Business Media, 2013.

[3] C. Xiao, H. Chen, Dimer Coverings on Random Polyomino Chains, *Zeitschrift für Naturforschung A*, Vol .70, No. 6, pp.465-470, June, 2015.

[4] S. Condamin, O. Bénichou, V. Tejedor, R. Voituriez, J. Klafter, First-passage Times in Complex Scale-invariant Media, *Nature,* Vol. 450, No. 7166, pp. 77-80, November, 2007.

[5] D. Aldous, Random Walks on Finite Groups and Rapidly Mixing Markov Chains, in: J. Azéma, M. Yor (Eds.), *Séminaire de Probabilités XVII 1981/82*, Springer, 1983, pp. 243-297.

[6] X.-S. Yang, S. Deb, Multiobjective Cuckoo Search for Design Optimization, *Computers & Operations Research*, Vol .40, No. 6, pp.1616-1624, June, 2013.

[7] B. Perozzi, R. Al-Rfou, S. Skiena, Deepwalk: Online Learning of Social Representations, *Proceedings of the 20th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (ACM)*, New York, NY, USA, 2014, pp. 701-710.

[8] F. Verstraete, M. M. Wolf, J. I. Cirac, Quantum Computation and Quantum-state Engineering Driven by Dissipation, *Nature physics,* Vol. 5, No. 9, pp. 633-636, July, 2009.

[9] R. Adhikari, R. K. Agrawal, A Combination of Artificial Neural Network and Random Walk Models for Financial Time Series Forecasting, *Neural Computing and Applications*, Vol. 24, No. 6, pp. 1441-1449, May, 2014.

[10] E. A. Codling, M. J. Plank, S. Benhamou, Random Walk Models in Biology, *Journal of the Royal Society Interface*, Vol. 5, No. 25, pp. 813-834, April, 2008.

[11] Q. Mei, D. Zhou, K. Church, Query Suggestion Using Hitting Time, *Proceedings of the 17th ACM conference on Information and knowledge management* (ACM), Napa Valley, CA, 2008, pp. 469-478.

[12] J. A. Byers, Correlated Random Walk Equations of Animal Dispersal Resolved By Simulation, *Ecology*, Vol. 82, No. 6, pp. 1680-1690, June, 2001.

[13] G. M. Viswanathan, E. P. Raposo, M. G. E. Da Luz, Lévy Flights and Superdiffusion in the Context of Biological Encounters and Random Searches, *Physics of Life Reviews*, Vol. 5, No. 3, pp. 133-150, September, 2008.

[14] V. Tejedor, O. Bénichou, R. Voituriez, Global Mean First-passage Times of Random Walks on Complex Networks, *Physical Review E*, Vol. 80, No. 6, 065104, pp. 1-4, December, 2009.

[15] J. Zhang, Y. Xiang, W. Sun, A Discrete Random Walk on the Hypercube, *Physica A: Statistical Mechanics and its Applications*, Vol. 494, pp. 1-7, March, 2018.

[16] M. S. Ansari, A. Mahani, Y. S. Kavian, Energy-Efficient Network Design via Modelling: Optimal Designing Point for Energy, Reliability, Coverage and End-to-end Delay, *IET Networks*, Vol. 2, No. 1, pp. 11-18, March, 2013.

[17] Z. Zhang, Y. Lin, S. Zhou, B. Wu, J. Guan, Mean First-passage Time for Random Walks on the T-graph, *New Journal of Physics*, Vol. 11, No. 10, 103043, pp. 1-13, October, 2009.

[18] F. Comellas, A. Miralles, Mean First-passage Time for

Random Walks on Generalized Deterministic Recursive Trees, *Physical Review E*, Vol. 81, No. 6, 061103, pp.1-12, June, 2010.

[19] R. Elsässer, T. Sauerwald, Tight Bounds for the Cover Time of Multiple Random Walks, *Theoretical Computer Science*, Vol. 412, No. 24, pp. 2623-2641, May, 2011.

[20] Z. Zhang, Y. Qi, S. Zhou, W. Xie, J. Guan, Exact Solution for Mean First-passage Time on a Pseudofractal Scale-free Web, *Physical Review E*, Vol. 79, No. 2, 021127, pp. 1-6, February, 2009.

[21] A. El-Amawy, S. Latifi, Properties and Performance of Folded Hypercubes, *IEEE Transactions on Parallel and Distributed Systems*, Vol. 2, No. 1, pp. 31-42, January, 1991.

[22] S. Y. Hsieh, C. Y. Tsai, C. A. Chen, Strong Diagnosability and Conditional Diagnosability of Multiprocessor Systems and Folded Hypercubes, *IEEE Transactions on Computers*, Vol. 62, No. 7, pp. 1472-1477, July, 2013.

[23] C. N. Kuo, H. H. Chou, N. W. Chang, S. Y. Hsieh, Fault-Tolerant Path Embedding in Folded Hypercubes with Both Node and Edge Faults, *Theoretical Computer Science*, Vol. 475, pp. 82-91, March, 2013.

[24] X. -R. Xu, N. Cao, Y. Zhang, L. -Q. Gao, X.- L. Peng, X.- H. Lin, Spectra of Laplacian Matrix of Folded Hypercube Q_{fn}, *Journal of Dalian University of Technology,* Vol. 53, No. 5, pp. 777-780, September, 2013.

[25] D. J. Klein, M. Randić, Resistance Distance*, Journal of mathematical chemistry*, Vol. 12, No. 1, pp. 81-95, December, 1993.

[26] H. -Y. Zhu, D. J. Klein, I. Lukovits, Extensions of the Wiener number, *Journal of Chemical Information and Computer Sciences*, Vol. 36, No.3, pp. 420-428, May, 1996.

[27] D. E. Knuth, *The Art of Computer Programming*, Vol. 1, Addison-Wesley, 1997.

[28] J. Liu, X. -F. Pan, Y. Wang, J. Cao, The Kirchhoff Index of Folded Hypercubes and Some Variant Networks, *Mathematical Problems in Engineering*, Vol. 2014, 380874, pp. 1-9, January, 2014.

[29] H. -X. Yang, W. -X. Wang, Y. -C. Lai, Y. -B. Xie, B. -H. Wang, Control of Epidemic Spreading on Complex Networks by Local Traffic Dynamics, *Physical Review E,* Vol. 84, No. 4, 045101, pp. 1-4, October, 2011.

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