Convergence of Newton-relaxed Non-stationary Multisplitting Multi-parameters Methods for Nonlinear Equations

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Abstract

Based on the ideas [The Newton-parallel matrix multisplitting algorithms for solving systems of nonlinear equations, Journal of Sichuan Normal University (Natural Science), 1995, 18(4): 51-55] and [Global relaxed non-stationary multisplitting multi-parameters methods, International Journal of Computer Mathematics, 2008, 85(2): 211-224], this paper extends global relaxed multisplitting TOR iterative method for solving linear systems to that for nonlinear systems and presents Newton-relaxed non-stationary multisplitting multi-parameters USAOR method for nonlinear equations. Moreover, we also study the convergence of our methods, set up the convergence theorems and estimate the rate of convergence.

Keywords: Nonlinear equations, Matrix multisplitting method, Global relaxed multisplitting multiparameters method, H-matrix

1 Introduction

Consider the following nonlinear equations [36-37]

$$F(x)=0, F:\Omega\subset R^N\rightarrow R^N \quad (1)$$

where F is nonlinear mapping, Ω is any bounded set on R^N, and x is a real vector on Ω.

With the rapid development of technology, solution of nonlinear equation is more and more important. Nonlinear problems are of interest to engineers, physicists, mathematicians, and many other scientists because most systems are inherently nonlinear in nature. The application of scientific computing has been widely used in all walks of life, such as the analysis images of meteorological data, the shape design of aircraft, cars and ships, and the scientific calculation of high-tech research. Therefore, it is often necessary to find the root of the nonlinear equations (1). Many scholars have done a lot of research work on the numerical solution of the smooth nonlinear equations (1), and a series of efficient calculation methods have been obtained, which can be seen in [2-3, 6, 9-10, 13, 17, 22]. Even so, the algorithm design and theoretical analysis for the nonlinear equations are still far less mature and profound than the linear equations. For most iterative method for solving nonlinear equations, the structural idea comes from the iterative method of solving linear equations. The concept of multisplitting for the parallel solution of linear system was introduced by O'Leary and White [35] and further studied by many other authors [2-13, 15-17, 22-23, 26, 28-31, 34, 38, 40-50]. Among them, Prof. Bai, Huang and Gu et al. [2-13, 29-31] did great work. In 1993, Bai and Wang [2] set up a general framework of parallel matrix mullisplitting relaxation methods for solving large scale system of linear equations, investigated the convergence properties of this framework and gave several sufficient conditions. In 1994, Bai [3] gave comparisons of the convergence and divergence rates of the parallel matrix multisplitting iteration methods when the coefficient matrices are an L-matrix. In [31], Huang et al. further studied GRPM-style methods and gave comparisons of convergent and divergent rates for an H-matrix. In 1995, Bai [4-5] further discussed the convergence and divergence rates for a class of generalized matrix multisplitting relaxation methods in a detailed manner and revealed the inner links between two known frameworks of multisplitting relaxation methods. In 1996, Bai et al. [6-7] constructed various synchronous and asynchronous parallel matrix multisplitting iterative methods suitable to the SIMD and MIMD multiprocessor systems and set up a class of parallel nonlinear AOR method in the sense of matrix multi-splitting for solving the large scale system of nonlinear equations. In [12-13, 34], Bai et al. studied the nonstationary multisplitting iteration methods and the nonstationary multisplitting two-stage iteration methods when the coefficient matrix is an H-matrix or a positive definite matrix or a hermitian positive

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DOI: 10.3966/160792642019052003015

definite matrix. In [29-30], Gu et al. further studied relaxed nonstationary two-stage matrix multisplitting methods and the corresponding asynchronous schemes. In [16], Cao studied the convergence of nested stationary iterative methods when the coefficient matrix is monotone including H -matrices. In [26], Evans, Wang and Bai proposed a class of matrix multisplitting multiparameter relaxation methods including the matrixmultisplitting SOR method, the extrapolated matrix multisplitting AOR method, the matrix multisplitting SSOR and SAOR methods as its special cases for solving large nonsingular systems of equations and established the convergence theory of this new class of methods under the condition that the coefficient matrix of the system of equations is an H -matrix. In [40-41, 43-44, 46-47], Wang et al. further studied the convergence of relaxed parallel multisplitting AOR, USAOR and SSOR methods for an H -matrix. In [22-23, 45], Chang and Zhang et al. further proposed the parallel multisplitting TOR method for solving a large nonsingular systems of linear equations when the coefficient matrices are an H -matrix and established the convergence theorem of this new algorithm. Then, Zhang et al. [48] further studied the parallel multisplitting TOR method and obtained the better convergence results. In [17, 38], Cao et al. analyzed the convergence of two different variants of relaxed multisplitting methods with different weighting schemes when A is a monotone matrix or an M -matrix. In [49], Zhang et al. studied the non-stationary matrix multisplitting multiparameters methods for almost linear systems when the matrix A is a square nonsingular H -matrix. Using the similar ideas, Zhang et al. [50] analyzed modulus-based synchronous multisplitting multi-parameters methods for linear complementarity problems when the system matrix is an H_+ -matrix. In [33], Li extended the corresponding methods to the solution of nonlinear equations, and constructed and the Newton-parallel multisplitting algorithms. On the basis, this paper extends relaxed parallel multisplitting USAOR iterative method for solving linear systems to that for nonlinear systems. Moreover, we study the convergence of our methods, set up the convergence theorems and estimate the rate of convergence. When choosing the approximately optimal relaxed parameters, Newton-relaxed parallel multisplitting multi-parameters USAOR iterative methods presented in this paper for nonlinear systems will have faster convergence rate than other methods.

The Newton method for solving equations (1) is as follows:

$$x^{k+1} = x^k - F'(x^k)^{-1} F(x^k), k = 0,1,..., \quad \textbf{(2)}$$

The corresponding Newton equations are

$$F'(x^k)x = F'(x^k)x^k - F(x^k). \quad \textbf{(3)}$$

For convenience, define

$$A(x^k) = F'(x^k), b(x^k) = F'(x^k)x^k - F(x^k), \quad \textbf{(4)}$$

Then the formula (3) can be expressed as

$$A(x^k)x = b(x^k), k = 0,1... \quad \textbf{(5)}$$

On the basic knowledge of nonlinear equations (1), please refer to the literature [15, 30]. If the case of relaxed matrix multisplitting is applied, then we can obtain Newton-relaxed parallel multisplitting iterative method, abbreviated as NRPM iterative method.

Algorithm 1. (Relaxed Parallel Multisplitting Method)

Given the initial vector $x^{(0)} \in R^N$,

For $m = 0,1,...,$ repeat (I) and (II), until convergence.

(I) For $t = 1,2,...,\alpha,$ (parallel) solving y_t :

$$M_t(x^k)y_t = N_t(x^k)x^m + b(x^k) \text{ (Local relaxation)} \quad \textbf{(6)}$$

(II) Computing

$$x^{m+1} = \gamma \sum_{t=1}^{\alpha} E_t y_t + (1-\gamma)x^m \text{ (System relaxation)} \quad \textbf{(7)}$$

Remark 1.1. If a diagonal element of the E_t is zero, the corresponding component of the y_t does not need be calculated, it greatly saves the amount of work. Therefore, this shows that E_t also play a role in the allocation of the workload of the processor. We should choose E_t to load the balance between the processors as far as possible, thereby reduce the cost of waiting for synchronization. Obviously, Algorithm 1 has the nature of parallelism.

Lemma 1.1. Let A be an H -matrix, then A is nonsingular, and $|A|^{-1} \le \langle A \rangle^{-1}$.

Lemma 1.2. Let A and B be M -matrices. If $A \le B$, *then* $A^{-1} \ge B^{-1}$.

Lemma 1.3. Let A be an H -matrix, and $A = D - B$, $D = diag(A)$, *then* $\rho(|D|^{-1}|B|) < 1$. Moreover, D is nonsingular.

2 Newton-relaxed Multisplitting USAOR Method

In order to improve the convergence speed, in this section we will design the Newton-relaxed parallel multisplitting USAOR Method. From multisplitting iteration scheme, we have

$$H(x^k) = I - G(x^k)A(x^k),$$

Then

$$x^{m+1} = (I - G(x^k)A(x^k))x^m + G(x^k)b(x^k), m = 0,1,2,...,$$

To extense the iterative scheme, we will introduce

$$G(x^k)=\sum_{t=1}^{\alpha}E_tM_t(x^k)^{-1}, H(x^k)=I-G(x^k)A(x^k),$$

The corresponding iterative scheme is

$$x^{m+1}=H(x^k)x^m+G(x^k)b(x^k), k=0,1,2,..., \quad \textbf{(8)}$$

Define

$$A(x^k)=D(x^k)-L_t(x^k)-U_t(x^k), t=1,2,...,\alpha,$$

where $D(x^k)=diag(A(x^k)), L_t(x^k)$ is strictly lower triangular matrix, $U_t(x^k)$ is strictly upper triangular matrix, satisfying

$$U_t(x^k)=D(x^k)-L_t(x^k)-A(x^k)$$

Then $(D(x^k)-L_t(x^k), U_t(x^k), E_t)$ is a multisplitting of $A(x^k)$.

Algorithm 2. (Relaxed Non-stationary Parallel Multisplitting Method)

Given the initial vector $x^{(0)}\in R^N$,

For $m=0,1,...,$ repeat (I) and (II), until convergence.

For $t=1,2,...,\alpha$, Define $y_t^{(0)}=x^m$.

(I) For $i=1,2,...,q(m,t)$ (parallel) solving y_t^i :

$$M_t(x^k)y_t^i=N_t(x^k)y_t^{(i-1)}+b_i(x^k) \quad \textbf{(9)}$$

(II) Computing

$$x^{m+1}=\gamma\sum_{t=1}^{\alpha}E_ty_t^{q(m,t)}+(1-\gamma)x^m \quad \textbf{(10)}$$

If applying USAOR iteration method on Algorithm 2, we may obtain relaxed nonstationary multisplitting multi-parameters USAOR method, denoted as RNMM-USAOR method, whose iteration format is as follows:

$$x^m=H_{RNMM-USAOR}x^m+\gamma G_{RNMM-USAOR}b(x^k), \quad \textbf{(11)}$$

where

$$H_{RNMM-USAOR}=\gamma H+(1-\gamma)I,$$

$$H=\sum_{t=1}^{\alpha}E_t(Q_{\eta,\xi},P_{\alpha,\beta})^{q(m,k)}$$

$$G_{RNMM-USAOR}=\sum_{t=1}^{\alpha}K_t\{\sum_{i=1}^{q(m,k)-1}[W_\beta V_\xi]^{-1}[R_{\eta,\xi}M_{\alpha,\beta}]^i\}[W_\beta V_\xi]^{-1}\eta_t\alpha_t,$$

$$Q_{\eta,\xi}=(D(x^k)-\xi_tL_t(x^k))^{-1}[(1-\eta_t)D(x^k)+(\eta_t-\xi_t)L_t(x^k)+\eta_tU_t(x^k)]=V_\xi^{-1}R_{\eta,\xi},$$

$$P_{\alpha,\beta}=(D(x^k)-\beta_tU_t(x^k))^{-1}[(1-\alpha_t)D(x^k)+(\alpha_t-\beta_t)U_t(x^k)+\alpha_tL_t(x^k)]=W_\beta^{-1}M_{\alpha,\beta}. \quad \textbf{(12)}$$

Obviously, the iterative scheme (11) converges if and only if $\rho(H_{RNMM-USAOR})<1$. If we approximately solve Newton equations (5) by using RNMM-USAOR method, then we can obtain Newton-relaxed non-stationary multisplitting multi-parameters USAOR method, denoted as NRNMM-USAOR method, whose iteration format is as follows:

$$\xi^{(1)}: x^{k+1}=x^k-[H_{NRNMM-USAOR}(x^k)^{l-1}+......+I]G_{NRNMM-USAOR}(x^k)F(x^k) \quad \textbf{(13)}$$

Based on the iterative scheme (13), when choosing different parameters $\alpha_k,\beta_k,\eta_k,\xi_k,\gamma$ and $q(m,t)$, we will obtain different iterative scheme. Please refer to Table 1.

Table 1. Diffetent Newton-relaxed multisplitting iterative scheme

$\alpha_k,\beta_k,\eta_k,\gamma,q(m,t)$	Method	Ref
$\alpha_1,\beta_1,\alpha_1,\beta_1,1,1$	NM-USSOR	this paper
$\alpha_1,\beta_1,\alpha_2,\beta_2,1,1$	NM-USSAOR	this paper
$\alpha_1,\beta_1,\alpha_1,\beta_1,\gamma,1$	NRM-USSOR	this paper
$\alpha_1,\beta_1,\alpha_2,\beta_2,\gamma,1$	NRM-USAOR	this paper
$\alpha_t,\beta_t,\alpha_t,\beta_t,1,q(m,t)$	NNMM-USSOR	this paper
$\alpha_t,\beta_t,\eta_t,\xi_t,1,q(m,t)$	NNMM-USAOR	this paper
$\alpha_t,\beta_t,\alpha_t,\beta_t,\gamma,q(m,t)$	NRNMM-USSOR	this paper
$\alpha_t,\beta_t,\eta_t,\xi_t,\gamma,q(m,t)$	NRNMM-USAOR	this paper

Remark 2.1. NRNMM-USAOR method uses more relaxed factors and is the generalization of a variety of Newton-multisplitting iterative scheme. Moreover, each processor can select different parameters, so it is suitable for parallel computing. For NRNMM-USAOR method, we can choose proper E_t in order to make each processor achieve load balance and avoid waiting for synchronization. Moreover, if we can select appropriate relaxed factors, then the convergence speed may be improved and solving time may be greatly reduced.

3 Convergence Analysis

Lemma 3.1. [33] Assume F is G-differentiable in an open neighborhood $\Omega_0\subset\Omega$ of $x^*\in\text{int}(\Omega)$, F' is continuous in x^* and $F'(x^*)$ is nonsingular, such that $F'(x^*)=0(t=1,2,...,\alpha)$ $(M_t(x^*),N_t(x^*),E_t)$ is a multisplitting of $F'(x^*)$. If $M_t:\Omega_0\to R^{N\times N}$ is the attraction [27] of iterative scheme ξ, $\forall$ positive integer $l>1$, and $R_1(\xi,x^*)=\rho\left[H(x^*)^l\right]$, where $R_1(\xi,x^*)$ is R-factor [27] of iterative scheme ξ in $x^*,\rho(G)$ is the spectral radius of matrix G.

Theorem 3.2. Assume F is G-differentiable in an open neighborhood $\Omega_0 \subset \Omega$ of $x^* \in \text{int}(\Omega), F'$ F' is continuous in x^* and $F'(x^*)$ is nonsingular. Let $F'(x) = D(x) - L_t(x) - U_t(x), t = 1,2,...,\alpha$ is the decomposition of the iterative scheme (11), $L_t(x)$ is continuous in x^* and $D(x^*) = diag(F'(x^*))$ is nonsingular, $\sum_{t=1}^{\alpha} E_t = I, E_t$ is diagonal non-negative matrix. Consider Newton-relaxed non-stationary multisplitting Multi-parameters USAOR method

$$x^{k+1} = x^k - \left[H_{GRNMM-USAOR}\left(x^k\right)^{l-1} + ... + I\right]$$
$$G_{GRNMM-USAOR}\left(x^k\right)F\left(x^k\right), k = 0,1,...$$

where $H_{RNMM-USAOR}, G_{RNMM-USAOR}$ is defined before. If $\rho\left[H_{RNMM-USAOR}\left(x^k\right)\right] < 1$, then x^* is the attration [23] of iterative scheme $\xi^{(1)}$, and $R_1\left(\xi^{(1)}, x^*\right) = \rho\left[H_{RNMM-USAOR}\left(x^*\right)\right]$, where $R_1\left(\xi^{(1)}, x^*\right)$ is R-factor [27] of iterative scheme $\xi^{(1)}$ in x^*.

Proof. From hypothesis, we can find: $F'(x) = M_t(x) - N_t(x), t = 1,2,...,\alpha.$ Since $F', L_t(x)$ is continuous in x^*, then $M_t(x)$ is also continuous in $x^*, t = 1,2,...,\alpha$. Since $L_t(x^*)$ is strictly low triangular matrix and $D(x^*)$ is nonsingular, then we can obtain that $M_t(x^*)$ is also nonsingular, $t = 1,2,...,\alpha$. It is easy to meet all the conditions of Lemma 3.1. By Lemma 3.1, Theorem 3.2 is established.

Theorem 3.3. Assume the conditions are satisfied of F and x^* in Theorem 3.2. $F'(x^*) = D(x^*) - L_t(x^*) - U_t(x^*)$ is H-matrix, $\left(D(x^*) - L_t(x^*), U_t(x^*), E_t\right)(t = 1,2,...,\alpha)$ is multisplitting of $F'(x^*)$. Let $\langle F'(x^*)\rangle = |D(x^*)| - |L_t(x^*)| - |U_t(x^*)| = |D(x^*)| - |B(x^*)|$. If $0 < \alpha_t, \eta_t < \frac{2}{1+\rho}, 0 \le \beta_t \le \alpha_t, 0 \le \xi_t \le \eta_t, 0 < \gamma < \frac{2}{1+\rho'}$, then x^* is the attration [27] of iterative scheme $\xi^{(1)}$, and $\rho = \rho\left(|D(x^*)|^{-1}|B(x^*)|\right) = \rho(J)$, $R_1\left(\xi^{(1)}, x^*\right) = \rho\left[H_{RNMM-USAOR}\left(x^*\right)\right]$, $\rho' = \max_{1 \le t \le \alpha}\{|1-\alpha_t| + \alpha_t\rho, |1-\eta_t| + \eta_t\rho\}, q(m,t) \ge 1$, $m = 0,1,2...$, $t = 1,2,...\alpha$.

Proof. Since $\rho\left[H_{GRNMM-USAOR}\left(x^*\right)\right] \le \rho\left[|H_{GRNMM-USAOR}\left(x^*\right)|\right]$, we only prove that $\rho\left[|H_{GRNMM-USAOR}\left(x^*\right)|\right] < 1$, then we can obtain $\rho\left[H_{GRNMM-USAOR}\left(x^*\right)\right]$. By Theorem 3.2 we can see that this theorem is established.

At first, we will prove $|H|x \le \theta x$ $(m = 0,1,2.....,0 \le \theta < 1)$.

By hypothetical conditions, we know $D(x^*) - \beta_t U_t(x^*)$ and $D(x^*) - \xi_t L_t(x^*)$ are both H-matrix, $k = 1,2,3....\alpha$. By Lemma 1.1 and definition of comparison matrix, it is not difficult to find

$$\begin{aligned} &|(D(x^*) - \beta_t U_t(x^*))^{-1}| \le (D(x^*) - \beta_t U_t(x^*))^{-1} \\ &= (|D(x^*)| - \beta_t |U_t(x^*)|)^{-1}. \\ &|(D(x^*) - \xi_t L_t(x^*))^{-1}| \le (D(x^*) - \xi_t L_t(x^*))^{-1} \\ &= (|D(x^*)| - \xi_t |L_t(x^*)|)^{-1}. \end{aligned} \quad \textbf{(14)}$$

Case 1: when $0 < \alpha_t, \eta_t \le 1, 0 \le \beta_t \le \alpha_t, 0 \le \xi_t \le \eta_t$, $0 < \gamma < \frac{2}{1+\rho'}$ Define

$$\begin{aligned} &\tilde{M}_t^1(x^*) = |D(x^*)| - \beta_t |U_t(x^*)|, \tilde{M}_t^2(x^*) \\ &= |D(x^*)| - \xi_t |L_t(x^*)|. \\ &\tilde{N}_t^1(x^*) = (1-\alpha_t)|D(x^*)| + (\alpha_t - \beta_t)|U_t(x^*)| + \alpha_t |L_t(x^*)| \\ &= \tilde{M}_t^1(x^*) - \alpha_t(|D(x^*)| - |B(x^*)|). \\ &\tilde{N}_t^2(x^*) = (1-\eta_t)|D(x^*)| + (\eta_t - \xi_t)|L_t(x^*)| + \eta_t |U_t(x^*)| \\ &= \tilde{M}_t^2(x^*) - \eta_t(|D(x^*)| - |B(x^*)|). \end{aligned} \quad \textbf{(15)}$$

By equation (11), we can get

$$\begin{aligned} &|P_{\alpha,\beta}(x^*)| \le (\tilde{M}_t^1(x^*))^{-1}\tilde{N}_t^1(x^*) \\ &= (\tilde{M}_t^1(x^*))^{-1}[(\tilde{M}_t^1(x^*))^{-1} - \alpha_t(|D(x^*)| - |B(x^*)|)] \\ &= I - \alpha_t(\tilde{M}_t^1(x^*))^{-1}|D(x^*)|(I - |D(x^*)|^{-1}|B(x^*)|). \end{aligned} \quad \textbf{(16)}$$

Let e denote the vector $e = (1,1,1,.....1)^T \in R^N$. Since J is nonnegative, the matrix $J + \varepsilon ee^T$ has only positive entries and irreducible for any $\varepsilon > 0$. By the Perron-Frobenius theorem for any $\varepsilon > 0$, there is a vector $x_\varepsilon > 0$ such that

$$(J + \varepsilon ee^T)x_\varepsilon = J_\varepsilon x_\varepsilon = \rho_\varepsilon x_\varepsilon,$$

where $\rho_\varepsilon = \rho(J + \varepsilon ee^T) = \rho(J_\varepsilon)$. Moreover, if $\varepsilon > 0$ is small enough, we have $\rho_\varepsilon < 1$ by continuity of the spectral radius. Because of $0 < \alpha_t \le 1$, we also have $1 - \alpha_t + \alpha_t\rho_\varepsilon < 1$, and $1 - \alpha_t + \alpha_t\rho < 1$. Since $(\tilde{M}(x^*))^{-1} \ge |D(x^*)|^{-1}$, we have

$$\begin{aligned}|P_{\alpha,\beta}(x^*)| &\le I-\alpha_t\,|D(x^*)|^{-1}|D(x^*)| \\ &\quad [|D(x^*)|^{-1}|B(x^*)|+eee^T)] \\ &= I-\alpha_t\,|D(x^*)|^{-1}|D(x^*)|[I-J_\varepsilon] \\ &= I-\alpha_t I+\alpha_t J_\varepsilon.\end{aligned} \tag{17}$$

and

$$\begin{aligned}|Q_{\eta,\beta}(x^*)| &\le I-\eta_t(\tilde{M}_t^2(x^*))^{-1}|D(x^*)|[I-J_\varepsilon] \\ &\le I-\eta_t I+\eta_t J_\varepsilon.\end{aligned}$$

Then, we can obtain

$$\begin{aligned}|H(x^*)|x_\varepsilon &\le |\sum_{t=1}^{\alpha}E_t(Q_{\xi,\eta}(x^*)P_{\alpha,\beta}(x^*))^{q(m,t)}|x_\varepsilon \\ &\le \sum_{t=1}^{\alpha}E_t[|Q_{\xi,\eta}(x^*)||P_{\alpha,\beta}(x^*)|]^{q(m,t)}x_\varepsilon \\ &\le \sum_{t=1}^{\alpha}E_t[(I-\eta_t I+\eta_t J_\varepsilon)(I-\alpha_t I+\alpha_t J_\varepsilon)]^{q(m,t)}x_\varepsilon \\ &= \sum_{t=1}^{\alpha}E_t[(I-\eta_t I+\eta_t J_\varepsilon)(I-\alpha_t+\alpha_t\rho_\varepsilon)]^{q(m,t)}x_\varepsilon \\ &= \sum_{t=1}^{\alpha}E_t[(I-\alpha_t+\alpha_t\rho_\varepsilon)(I-\eta_t+\eta_t\rho_\varepsilon)]^{q(m,t)}x_\varepsilon \\ &\le (1-\alpha_t+\alpha_t\rho_\varepsilon)(1-\eta_t+\eta_t\rho_\varepsilon)x_\varepsilon \\ &= \theta_1 x_\varepsilon\,(\varepsilon\to 0),\end{aligned} \tag{18}$$

where $\theta_1=(1-\alpha_t+\alpha_t\rho_\varepsilon)(1-\eta_t+\eta_t\rho_\varepsilon)<1$. Then, we have $|H(x^*)|x_\varepsilon \le \theta_1 x_\varepsilon$. and $\rho(H(x^*))\le\rho(|H(x^*)|)\le\theta_1<1$.

By similar proving process above, we have

$$\begin{aligned}|H_{RNMM-USAOR}(x^*)|x_\varepsilon &\le [|\gamma-1|+\gamma(1-\eta_t+\eta_t\rho_\varepsilon)]\times \\ &\quad [|\gamma\text{-}1|+\gamma(1-\alpha_t+\alpha_t\rho_\varepsilon)]x_\varepsilon \\ &\le [|\gamma-1|+\gamma\rho')][|\gamma-1|+\gamma\rho')]x_\varepsilon \\ &= \theta_2 x_\varepsilon.\end{aligned} \tag{19}$$

where $\theta_2=[|\gamma-1|+\gamma\rho')][|\gamma-1|+\gamma\rho')]<1$ and $\rho'=\max_{1\le t\le\alpha}\{|1-\alpha_t|+\alpha_t\rho,|1-\eta_t|+\eta_t\rho\}$.

Finally, we can obtain

$$|H_{RNMM-USAOR}(x^*)|x_\varepsilon \le \theta_2 x_\varepsilon.$$

and

$$\rho(H_{RNMM-USAOR}(x^*))\le\rho(|H_{RNMM-USAOR}(x^*)|)\le\theta_2<1.$$

Case 2: when $1<\alpha_t,\eta_t<\dfrac{2}{1+\rho}, 0\le\beta_t\le\alpha_t, 0\le\xi_t\le\eta_t$,

$0<\gamma<\dfrac{2}{1+\rho'}$,

Define

$$\begin{aligned}\tilde{N}_t^3(x^*) &= (\alpha_t-1)|D(x^*)|+(\alpha_t-\beta_t)|U_t(x^*)|+\alpha_t|L_t(x^*)| \\ &= \tilde{M}_t^1(x^*)-[(2-\alpha_t)|D(x^*)|-\alpha_t|B(x^*)|]. \\ \tilde{N}_t^4(x^*) &= (\eta_t-1)|D(x^*)|+(\eta_t-\xi_t)|L_t(x^*)|+\eta_t|U_t(x^*)| \\ &= \tilde{M}_t^2(x^*)-[(2-\eta_t)|D(x^*)|-\eta_t|B(x^*)|].\end{aligned} \tag{20}$$

So

$$\begin{aligned}|P_{\alpha,\beta}(x^*)| &\le (\tilde{M}_t^1(x^*))^{-1}\tilde{N}_t^3(x^*) \\ &= (\tilde{M}_t^1(x^*))^{-1}\{\tilde{M}_t^1(x^*)-[(2-\alpha_t)|D(x^*)|-\alpha_t|B(x^*)|]\} \\ &= I-(\tilde{M}_t^1(x^*))^{-1}|D(x^*)|[(2-\alpha_t)I-\alpha_t|D(x^*)|^{-1}|B(x^*)|] \\ &\le I-|D(x^*)|^{-1}|D(x^*)|[(2-\alpha_t)I-\alpha_t|D(x^*)|^{-1}|B(x^*)|] \\ &= I-[(2-\alpha_t)I-\alpha_t|D(x^*)|^{-1}|B(x^*)|] \\ &= (\alpha_t-1)I+\alpha_t|D(x^*)|^{-1}|B(x^*)|.\end{aligned} \tag{21}$$

and

$$\begin{aligned}|Q_{\xi,\eta}(x^*)| &\le I-[(2-\eta_t)I-\eta_t|D(x^*)|^{-1}|B(x^*)|] \\ &= (\eta_t-1)I+\eta_t|D(x^*)|^{-1}|B(x^*)|.\end{aligned}$$

Since $1<\alpha_t,\eta_t<\dfrac{2}{1+\rho}$, by similar proving process to Case 1, we have

$$\begin{aligned}|H(x^*)|x_\varepsilon &\le \sum_{t=1}^{\alpha}E_t[|Q_{\xi,\eta}(x^*)||P_{\alpha,\beta}(x^*)|]^{q(m,t)}x_\varepsilon \\ &\le \sum_{t=1}^{\alpha}E_t[(\eta_t I-I+\eta_t J_\varepsilon)(\alpha_t-I+\alpha_t J_\varepsilon)]^{q(m,t)}x_\varepsilon \\ &= \sum_{t=1}^{\alpha}E_t[(\eta_t I-I+\eta_t J_\varepsilon)(\alpha_t-1+\alpha_t\rho_\varepsilon)]^{q(m,t)}x_\varepsilon \\ &= \sum_{t=1}^{\alpha}E_t[(\alpha_t-1+\alpha_t\rho_\varepsilon)(\eta_t-1+\eta_t\rho_\varepsilon)]^{q(m,t)}x_\varepsilon \\ &\le (\alpha_t-1+\alpha_t\rho_\varepsilon)(\eta_t-1+\eta_t\rho_\varepsilon)x_\varepsilon \\ &= \theta_3 x_\varepsilon\,(\varepsilon\to 0),\end{aligned} \tag{22}$$

where $\theta_3=(\alpha_t-1+\alpha_t\rho_\varepsilon)(\eta_t-1+\eta_t\rho_\varepsilon)<1$. Then, we have $|H(x^*)|x_\varepsilon\le\theta_3 x_\varepsilon$, and $\rho(H(x^*))\le\rho(|H(x^*)|)\le\theta_3<1$.

By similar proving process above, we have

$$\begin{aligned}|H_{GRNMM-USAOR}(x^*)|x_\varepsilon &\le [|\gamma-1|+\gamma(\eta_t-1+\eta_t\rho_\varepsilon)]\times \\ &\quad [|\gamma-1|+\gamma(\alpha_t-1+\alpha_t\rho_\varepsilon)]x_\varepsilon \\ &\le [|\gamma-1|+\gamma\rho')][|\gamma-1|+\gamma\rho')]x_\varepsilon \\ &= \theta_4 x_\varepsilon.\end{aligned} \tag{23}$$

where $\theta_4=[|\gamma-1|+\gamma\rho')][|\gamma-1|+\gamma\rho')]<1$ and $\rho'=\max_{1\le t\le\alpha}\{|1-\alpha_t|+\alpha_t\rho,|1-\eta_t|+\eta_t\rho\}$. Finally, we can obtain

$$|H_{RNMM-USAOR}(x^*)|x_\varepsilon\le\theta_4 x_\varepsilon.$$

and

$$\rho(H_{GRNMM-USAOR}(x^*))\le\rho(|H_{GRNMM-USAOR}(x^*)|)\le\theta_4<1.$$

The Theorem is completed.

4 Numerical Experiments

In this section, we compare the previously established method in this paper by using the following example of the system of nonlinear equation (24). For solving nonlinear systems of the form

$$Ax + \phi(x) = b, \qquad \textbf{(24)}$$

where $A \in R^{n\times n}$ is a square nonsingular H-matrix, $x, b \in R^n$ and $\phi: R^n \to R^n$ is a nonlinear diagonal Mapping, i.e., the i th component ϕ_i of ϕ is a function Only of x_i, an iterative method is usually considered.

$$A = \begin{bmatrix} 8 & -1 & 0 & 0 & 0 & 0 \\ -1 & 8 & -1 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -1 & 8 & -1 \\ 0 & 0 & 0 & 0 & -1 & 8 \end{bmatrix} \in R^{n\times n},$$

$$D = diag(8, 8, ..., 8),$$

$$\phi(x) = \begin{bmatrix} x_1 e^{x_1} \\ x_2 e^{2x_2} \\ \vdots \\ x_{n-1} e^{(n-1)x_{n-1}} \\ x_n e^{nx_n} \end{bmatrix} : R^n \to R^n, b = \begin{bmatrix} 3 \\ 3 \\ \vdots \\ 3 \\ 3 \end{bmatrix} \in R^n.$$

We take $\alpha = 2$,

$$S_1 = \{1, 2, ..., m_1\}, S_2 = \{m_1, m_2 + 1, ..., n\},$$

with m_1, m_2 being positive integers satisfying $1 \le m_2 \le m_1 \le n$, and the particular multisplitting $(D - L_k, U_k, E_k), k = 1, 2$ of the coefficient matrix $A \in R^{n\times n}$.

$$L_k = (l_{ij}^{(k)}), l_{ij}^{(k)} = \begin{cases} 1, & if\ j = i-1, i, j \in S_k, \\ 0, & otherwise, \end{cases}$$

$$U_k = (u_{ij}^{(k)}), u_{ij}^{(k)} = \begin{cases} 0, & if\ j = i, \\ -(a_{ij} + l_{ij}^k), & otherwise, \end{cases}$$

$$e_j^{(1)} = \begin{cases} 1, \ if\ 1 \le j < m_2, \\ \frac{1}{2}, \ if\ m_2 \le j \le m_1, \\ 0, \ if\ m_1 < j \le n, \end{cases} \quad e_j^{(2)} = \begin{cases} 0, \ if\ 1 \le j < m_2, \\ \frac{1}{2}, \ if\ m_2 \le j \le m_1, \\ 1, \ if\ m_1 < j \le n, \end{cases}$$

The computations are proceeded with $m_1 = [\frac{4n}{5}]$, $m_2 = [\frac{n}{5}]$. Based on NRNMM-USAOR method, when $\xi_t = \beta_t, \eta_t = \alpha_t, \gamma = 1$, we will show the validity of Theorem 3.2 and Corollary 3.3 in this paper and compare the spectral radius of H with Theorem 3 in [33].

The above numerical examples not only clearly show the validity of Theorem 3.2 and Corollary 3.3 in this paper, but also show that the spectral radius of the iterative matrix with Theorem 3.3 is smaller than that in [33] when the same acceleration factor is adopted.

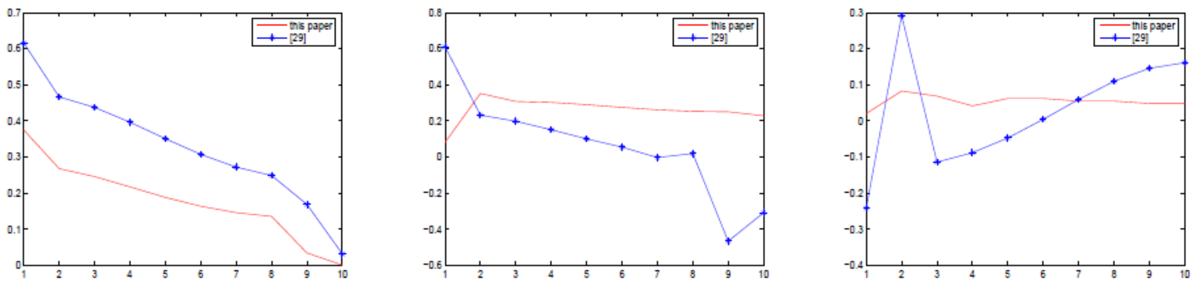

Figure 1. The eigenvalue distribution of the corresponding methods in this paper and [33] when $\xi_t, \beta_t, \eta_t, \alpha_t$ use the first three cases of Table 2, respectively. Here, $n = 10$.

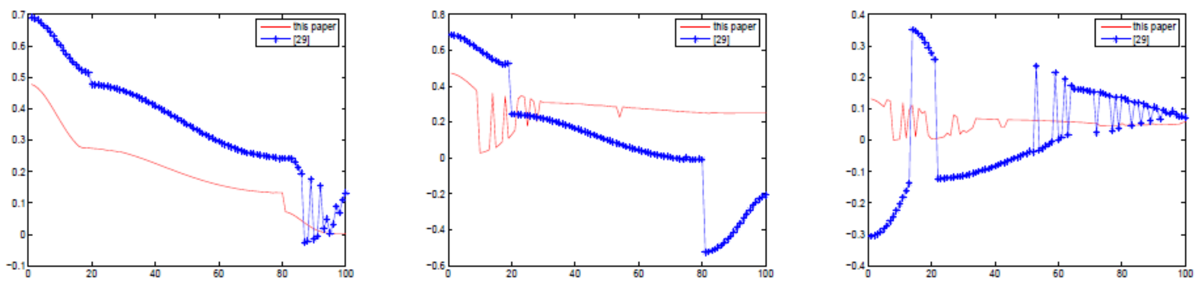

Figure 2. The eigenvalue distribution of the corresponding methods in this paper and [33] when $\xi_t, \beta_t, \eta_t, \alpha_t$ use the first three cases of Table 2, respectively. Here, $n = 100$.

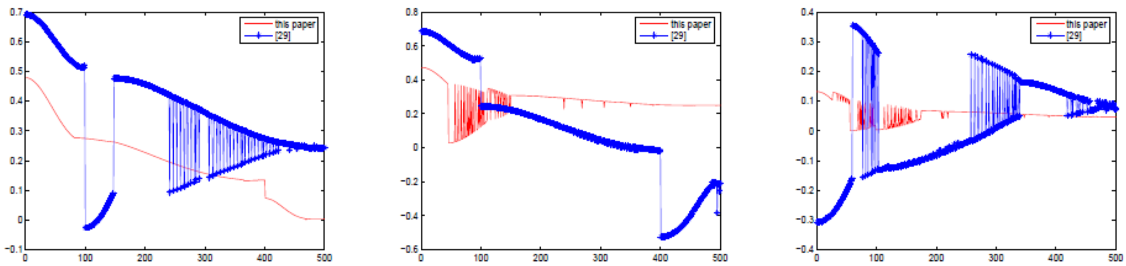

Figure 3. The eigenvalue distribution of the corresponding methods in this paper and [33] when $\xi_t, \beta_t, \eta_t, \alpha_t$ use the first three cases of Table 2, respectively. Here, $n = 500$.

Table 2. Comparisons of spectral radii of algorithms in this paper and [33]

Algorithum	n	$\frac{2}{1+e}$	$q(m,k)$	$\xi_t, \beta_t, \eta_t, \alpha_t$	$p(H)$
this paper	10	1.6131	1,1	$\xi_1 = \beta_1 = 0.2, \eta_1 = \alpha_1 = 0.4$ $\xi_2 = \beta_2 = 0.6, \eta_2 = \alpha_2 = 0.9$	0.3748
[33]	10	1.6131	1,1	$\xi_1 = 0.2, \eta_1 = 0.4, \xi_2 = 0.6, \eta_2 = 0.9$ $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$	0.6155
this paper	10	1.6131	1,1	$\xi_1 = \beta_1 = 0.3, \eta_1 = \alpha_1 = 0.4$ $\xi_2 = \beta_2 = 0.8, \eta_2 = \alpha_2 = 1.4$	0.3524
[33]	10	1.6131	1,1	$\xi_1 = 0.3, \eta_1 = 0.4, \xi_2 = 0.8, \eta_2 = 1.4$ $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$	0.6107
this paper	10	1.6131	1,1	$\xi_1 = \beta_1 = 0.8, \eta_1 = \alpha_1 = 1.2$ $\xi_2 = \beta_2 = 0.5, \eta_2 = \alpha_2 = 0.8$	0.0829
[33]	10	1.6131	1,1	$\xi_1 = 0.8, \eta_1 = 1.2, \xi_2 = 0.5, \eta_2 = 0.8$ $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$	0.2918
this paper	100	1.6002	1,1	$\xi_1 = \beta_1 = 0.2, \eta_1 = \alpha_1 = 0.4$ $\xi_2 = \beta_2 = 0.6, \eta_2 = \alpha_2 = 0.9$	0.4777
[33]	100	1.6002	1,1	$\xi_1 = 0.2, \eta_1 = 0.4, \xi_2 = 0.6, \eta_2 = 0.9$ $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$	0.6909
this paper	100	1.6002	1,1	$\xi_1 = \beta_1 = 0.3, \eta_1 = \alpha_1 = 0.4$ $\xi_2 = \beta_2 = 0.8, \eta_2 = \alpha_2 = 1.4$	0.4722
[33]	100	1.6002	1,1	$\xi_1 = 0.3, \eta_1 = 0.4, \xi_2 = 0.8, \eta_2 = 1.4$ $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$	0.6864
this paper	100	1.6002	1,1	$\xi_1 = \beta_1 = 0.8, \eta_1 = \alpha_1 = 1.2$ $\xi_2 = \beta_2 = 0.5, \eta_2 = \alpha_2 = 0.8$	0.1311
[33]	100	1.6002	1,1	$\xi_1 = 0.8, \eta_1 = 1.2, \xi_2 = 0.5, \eta_2 = 0.8$ $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$	0.3527
this paper	500	1.6000	1,1	$\xi_1 = \beta_1 = 0.2, \eta_1 = \alpha_1 = 0.4$ $\xi_2 = \beta_2 = 0.6, \eta_2 = \alpha_2 = 0.9$	0.4792
[33]	500	1.6000	1,1	$\xi_1 = 0.2, \eta_1 = 0.4, \xi_2 = 0.6, \eta_2 = 0.9$ $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$	0.6919
this paper	500	1.6000	1,1	$\xi_1 = \beta_1 = 0.3, \eta_1 = \alpha_1 = 0.4$ $\xi_2 = \beta_2 = 0.8, \eta_2 = \alpha_2 = 1.4$	0.4737
[33]	500	1.6000	1,1	$\xi_1 = 0.3, \eta_1 = 0.4, \xi_2 = 0.8, \eta_2 = 1.4$ $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$	0.6875
this paper	500	1.6000	1,1	$\xi_1 = \beta_1 = 0.8, \eta_1 = \alpha_1 = 1.2$ $\xi_2 = \beta_2 = 0.5, \eta_2 = \alpha_2 = 0.8$	0.1322
[33]	500	1.6000	1,1	$\xi_1 = 0.8, \eta_1 = 1.2, \xi_2 = 0.5, \eta_2 = 0.8$ $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$	0.3544

5 Conclusions

When the dimension of Newton equations is higher, the cost of the exact solution will be very large. Accordingly, the computation quantity of Newton method is also very large. In this paper, based on the literature [25-26, 33], by introducing a number of relaxed factors, we propose Newton-relaxed non-stationary multisplitting multi-parameters USAOR method, establish local convergence Theorem estimate and the convergence rate. Finally, numerical examples show the effectiveness the proposed method. For the iterative method proposed in this paper, the convergence speed is faster than other methods when selecting the approximately optimal parameters, we will study it in further work.

Acknowledgements

This research of this author is supported by the National Natural Science Foundation of China (11226337, 11501525), Excellent Youth Foundation of Science & Technology Innovation of Henan Province (184100510004), Science & Technology Innovation Talents in Universities of Henan Province (16HASTIT 040, 17HASTIT012), Aeronautical Science Foundation of China (2017ZD55014, 2016ZG55019), Project of Youth Backbone Teachers of Colleges and Universities of Henan Province (2015GGJS-003, 2015GGJS-179), Advanced Technological Research Project of Henan Province (182102110129), Henan Province Postdoctoral Science Foundation (2013031), Research on Innovation Ability Evaluation Index System and Evaluation Model (142400411268). The work of Chaoqian Li is supported partly by the Applied Basic Research Programs of Science and Technology Department of Yunnan Province [2018FB001], Outstanding Youth Cultivation Project for Yunnan Province [2018YDJQ021] and Program for Excellent Young Talents in Yunnan University. The work of Yaotang Li is supported by National Natural Science Foundation of China (11861077).

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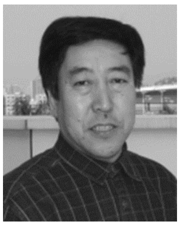

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